Quasi 4–8 Subdivision Surfaces

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Outline

- Subdivision Surfaces:  *Motivation and Background*
- 4–8 Meshes:  *Definition and Properties*
- Refinement Operators:  *Principles and Methods*
- Quasi 4–8 Meshes:  *Subdivision Scheme and Algorithm*
- Results:  *Analysis and Examples*
A **Subdivision Surface**, \( p \), is the limit surface generated by a **Subdivision Scheme**, \( S \), applied to a **control polygon**, \( p^0 \).

\[
p = \lim_{n \to \infty} S^n p^0
\]

- **Generalization of Spline Surfaces**
Motivation

● Advantages

  – Control Meshes with Arbitrary Topology
  – Global Smoothness and Local Features
  – Continuous Models from Discrete Representations
  – Simple and Efficient Algorithms
  – Natural Multiresolution Structure

● Disadvantages

  – May not have Closed Form Expression
Subdivision Schemes

- **Refinement:**
  - *Changes Mesh Topology to increase density*

- **Smoothing:**
  - *Changes Mesh Geometry to increase regularity*

\[ p^i \rightarrow \text{refinement} + \text{smoothing} \rightarrow p^{i+1} \]
Subdivision Operators

- Refinement Operator, $R$
  - Subdivision Template
    ($\odot$ new vertex, $\bullet$ old vertex)

- Smoothing Operator, $F$
  - Filter Mask (Stencil)
    $((c_i)_{i \in N(p)})$
    
    $p^{j+1} = \sum_{i \in N(p)} c_i p^j_i$

* Subdivision Operators depend on discrete vertex neighborhoods, $N(p)$
Mesh Structure

- Discrete Local Topology
  - 1-Neighborhood of a Vertex: $N_1(v)$
  - Valence (degree) of a vertex: $\text{deg}(v)$

- Classification
  - Regular Meshes
    (ordinary vertices)
  - Semi-regular Meshes
    (isolated extraordinary vertices)
Analysis of Subdivision Schemes

Order of Continuity, $C^k$, of the Limit Surface

- Convergence Properties: Study Invariant Neighborhoods

- Stationary Subdivision: Subdivision Matrix $S$
Quasi 4–8 Subdivision Surfaces

New Subdivision Scheme

- Based on Semi-regular 4-directional Meshes
- Generates Surfaces of class $C^1$
4–8 Meshes

- Regular 4–directional mesh
  - Generated by the set of vectors: \( \{e_1, e_2, e_1 + e_2, e_1 - e_2\} \)

* Regular vertices: valence 4 and 8

* Symmetry Properties: Leaves Tiling \([4.8^2]\)
Quasi 4–8 Meshes

- Semi-regular 4–direction Mesh

* Most Properties of 4–8 Meshes

* Generated by Refinement
Refinement of 4–8 Meshes

- Normal Refinement (quaternary)
  1. Split edges in 2
  2. Subdivide face in 4

- Interleaved Refinement (recursive binary)
  1. Split main edge
  2. Subdivide face in 2
  3. Refine subfaces
Quasi 4–8 Subdivision Scheme

- Overview
  1.a Find Maximal Independent Set of 2-Face Clusters
  1.b Perform Binary Subdivision on Faces

- Apply Smoothing Filter to Vertices

- Algorithm

\[
\text{quasi}_4\text{–}8\text{ subdivision } (K)
\]
\[
\text{quasi}_4\text{–}8\text{ refinement } (K)
\]
\[
\text{quasi}_4\text{–}8\text{ smoothing } (K)
\]
Refinement Operator

- Select Clusters based on Edge Length
  - Binary Subdivision Template
  - Internal Edge Bisection
Refinement Algorithm

quasi_4–8_refinement \((K)\)

sort_edges \((E)\)
store \(e \in E\) in priority queue \(Q\)

while \(Q \neq \emptyset\) do

get \(e\) from \(Q\)
if \(e\) not marked then
split \((e)\)
mark_cluster \((e)\)

for all \(f \in F\) do
subdiv \((f)\)
Smoothing Operator

- Smoothing Templates

New Vertices  Ordinary Vertices  Extraordinary Vertices
Smoothing Algorithm

quasi_4–8_smoothing \((K)\)

\[
\text{for } v'_i \in V' \text{ do}
\]

\[
p^{l+1}(v'_i) = \frac{1}{4} \sum_{v_j \in N_1(v'_i)} p^{l}(v_j)
\]

\[
\text{for } v_i \in V \text{ do}
\]

\[
p^{l+1}(v_i) = \frac{1}{2} p^{l}(v_i) + \frac{1}{2n} \sum_{v_j' \in N_1(v_i) \cap V'} p^{l+1}(v'_j)
\]

- Cascade Convolution

![Diagram of Cascade Convolution](image_url)
Convergence of the Scheme

- Quasi-Stationarity
  - Geometric reasoning based on planar case

- $C^1$ Continuity
  - Eigenanalysis of the Subdivision Matrix
  - Investigation of Characteristic Map
Planar Case

1. newly inserted vertices have valence 4.
2. old vertices of valence 4 change to valence 8.
3. old vertices of valence 8 do not change.
4. vertices of any other valence change little.
$C^1$ Continuity

- Invariant Neighborhood of Regular Vertex $v \in V_8$

- Subdivision Matrix $S_8$
  
  \[
  S_8 = \frac{1}{32} \begin{pmatrix}
    20 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
    8 & 8 & 0 & 0 & 0 & 8 & 0 & 0 \\
    8 & 0 & 8 & 0 & 0 & 8 & 0 & 0 \\
    8 & 0 & 0 & 8 & 0 & 8 & 0 & 0 \\
    8 & 0 & 0 & 0 & 8 & 0 & 8 & 0 \\
    12 & 2 & 2 & 0 & 0 & 12 & 2 & 0 \\
    12 & 0 & 2 & 2 & 0 & 2 & 12 & 2 \\
    12 & 0 & 0 & 2 & 2 & 0 & 2 & 12 \\
    12 & 2 & 0 & 0 & 2 & 2 & 0 & 12
  \end{pmatrix},
  \quad
  \lambda_i = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}
  
  1. $\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > |\lambda_3|$;

  2. eigenvector corresponding to $\lambda_0$ is $(1, 1, 1, 1, 1, 1, 1, 1)$;
Subdivision of N-Regular Planar Polygons

- Intuition for Characteristic Map Induced by $S_k$
Subdivision of Warped Hexagon

- Adaptive Refinement
Examples

- Simple Shapes
  - Cube: genus 0 surface
  - Torus: genus 1 surface

- Feature Control
  - Boundary and Creases

- Complex Objects
  - Bunny
  - Cow
Cube
Torus
Cylinder

- Feature Control: Tagged Edges
Bunny
Cow
Conclusions

- New Subdivision Scheme
  - Well Coupled Operations
  - Adapted Meshes
  - Simple and Efficient

- New Concepts
  - Geometric-Sensitive Refinement
  - Quasi Stationary Subdivision