Wavelets and Models

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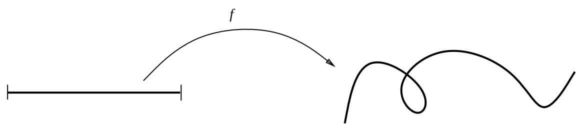
Outline

- Simple Parametric Models
- Arbitrary Topology
- Modeling Operations
- Implicit Models
- Conversion

Simple Parametric Models

- Parametric Model
  - Canonical Domain
  - Coordinate Functions
- Simple Models
  - Topology
  - Curves and Sheets
- Curves
  - $f : [0, 1] \to \mathbf{R}^2$
  - $f(u) = (x(u), y(u))$



- Sheets
  - Tensor Product
  - $f : [0, 1] \times [0, 1] \to \mathbf{R}^3$

Wavelet for Parametric Models

- Types of Wavelets
  - Wavelets on Interval
  - 2D: Tensor Product Wavelets
- Basis Functions
  - Scaling (Splines)
- Modeling Framework
  - Synthesis vs. Analysis
  - Discrete (knots) v.s Continuous (surfaces)
- Properties
  - Smoothness
  - Local Support
  - Compatibility with Standards
- Interpolation vs. Approximation
- *Matrix Notation*

Splines and Wavelets

- Scaling Functions

$$f^n(u) = \Phi^n(u)c^n$$

$$f = (\phi_0^n \cdots \phi_m^n) \begin{pmatrix} c_0^n \\ \vdots \\ c_m^n \end{pmatrix}$$

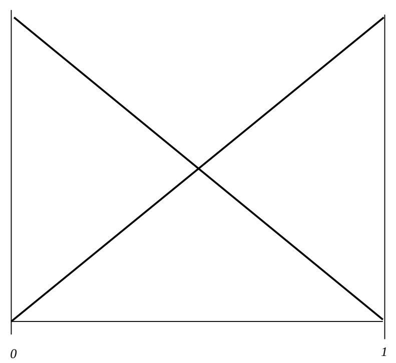
- Connection with Splines
  - Polynomial Bases

$$f(u) = \underbrace{uB}_{\Phi} \; c$$

- MRA on Intervals
  - Coarsest Level: V_0 ($\mu(I) < \infty$)
    - V^j, $j \in \mathbf{N}$
  - Partition of Unity
    - V_0^n
    - n degree of polynomial

Example

- Linear Splines



$$f_0^1(u) = \phi_{0,1}^1(u)c_1 + \phi_{0,2}^1(u)c_2$$

$$f_0^1(u) = (1-u)\,c_1 + u\,c_2$$

$$f(u) = (u \;\; 1)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$f(u) = (\phi_{0,1}^1 \;\; \phi_{0,2}^n)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Matrix Framework for Wavelets

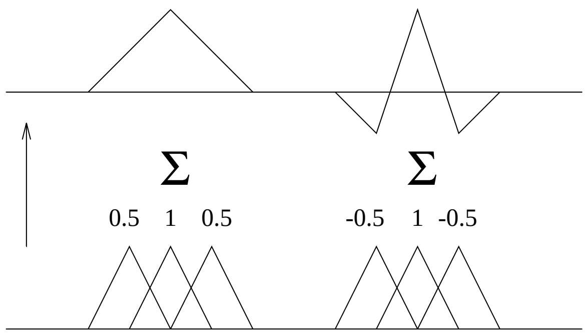
- Refinement

  $\Phi^{j-1} = \Phi^j P^j$

  $\Psi^{j-1} = \Phi^j Q^j$

  $(Phi^{j-1}|\Psi^{j-1}) = \Phi^j (P^j|Q^j)$

  - linear combination



- Inverse

  $(A^j B^j) = (P^j|Q^j)^{-1}$

  $(\Phi^{j-1}|\Psi^{j-1})(A^j B^j) = \Phi^j$

- Matrices

  A, B - Analysis

  P, Q - Synthesis

Computation

- Knot Vector

  $c^{\,n}$

- Spline Representation

  $f = \Phi^n c^n$

- Decomposition

  $c^{n-1} = A^n c^n$

  $d^{n-1} = B^n c^n$

- Reconstruction

  $c^n = P^n c^{n-1} + Q^n d^{n-1}$

- Levels

  j = n to 0

Surfaces with Arbitrary Topology

- Fundamental Domains
  - Compact without Borders
  - Genus: 0, 1, ...
- Wavelets on General Domains
  - Topology
  - Metric
    - Adapted to a Measure
- Wavelets on the Sphere
  - Important Case
  - Applications
    - GIS
      - Topography, Remote Sensing
    - Illumination Problems
      - BRDFs, Radiance
    - Texture
      - Environment Maps

Generalized Wavelets

- Differences from Classic Wavelets
  - Filter Coefficients Vary
    - Reflect Change in Measure
  - Many Wavelet Functions
    - Can't use Translation and Scaling
- Basic Notion
  - Basis written as Linear Combination
  - Neighborhood Analysis
- Construction
  - MRA
    - Subdivision Domains
  - Geometric Derivation
    - Sequence of Surfaces
  - Functions on $\mathbf{S}$
    - Displacement along Normal

Implementation

- Techniques
  - Pseudo Orthonormalization
  - Lifting Scheme
- Fundamental Element
  - Triangular Patch
- Data Structures
  - Triangle Quadtree
  - Flat Matrix
- Neighborhood Structure
  - Index Set

Operations with Models

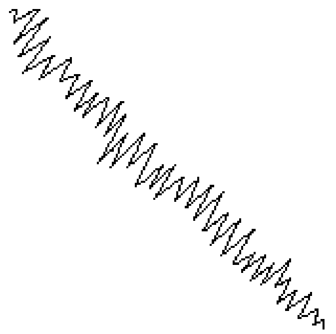
- Multiresolution Editing
  - Structure
  - Character
- Free Form Interaction
  - Variable Range
- Representation
  - Smoothing
  - Compression
- Display
  - Progressive
  - Adapted

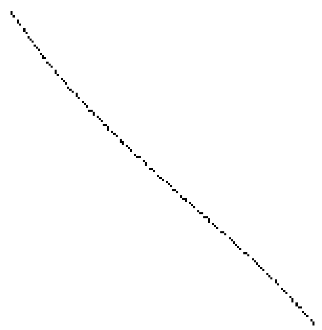
Multiresolution Editing

$$f^n = \Phi^l c^l + \Psi^m d^m$$

where $n = l + m$

- Two Components
  - Smooth Structure
    $\Phi^l c^l$
  - Modulation
    $\Psi^m d^m$

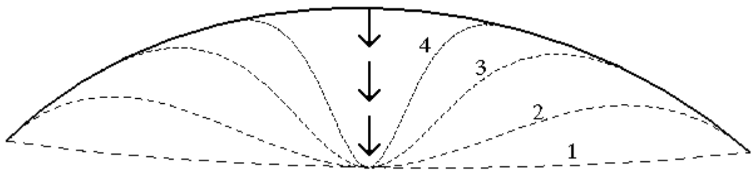




Multiresolution Editing

- Integer Level Editing

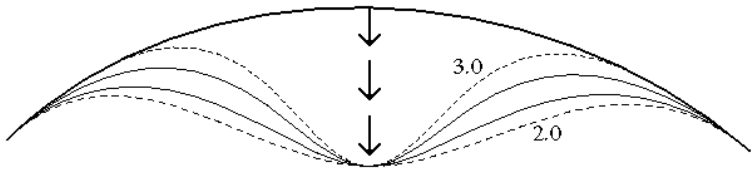
  $\widetilde{c}^j = c^j + \Delta c^j$



- Fractional Level Editing

  $f^{j+t}(u) = (1-t)f^j(u) + tf^{j+1}(u)$

  $\widetilde{c}^{j+t} = (1-t)P^{j+1}c^j + tc^{j+1}$



Editing the Structure

- Original Knots at Level j

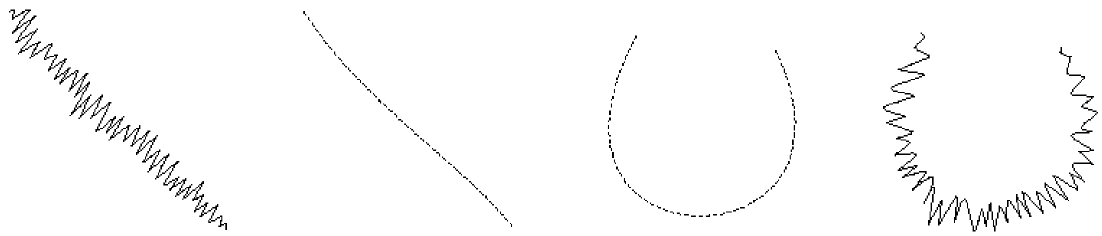
  c^j

- Edited Knots

  $\hat{c}^j = c^j + \Delta c^j$

- New Model at High Resolution

$$\begin{aligned} \hat{c}^n &= c^n + \Delta c^n \\ &= c^n P^n P^{n-1} \ldots P^{j+1} \Delta c^j \end{aligned}$$



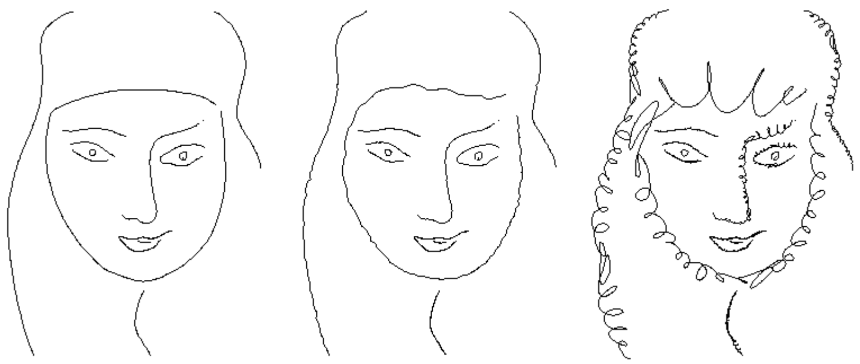
(a) (b) (c) (d)

Editing the Character

- Range of Detail
  $r = [j, n]$

- Computation
  - Replace
    $\{d^j, \ldots d^{n-1}\}$ with $\{\hat{d}^j, \ldots \hat{d}^{n-1}\}$
  - Reconstruct

- Character Library
  - Samples
  - Procedural Displacement Functions

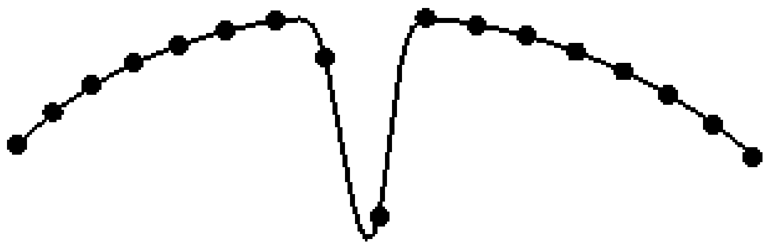


Direct Manipulation

- Points on the Model vs. Knots: $f(u_0)$
- Edit at Arbitrary Level: $f^{j+t}(u_0)$
- Displacements: $f^{j+t}(u_0) + \delta$
- Computation
  - Given $f(u_0)$, find u_0
  - Compute changes to c^{j+t} near u_0
- Finding u_0
  - Scan conversion (curves)
- Changes to c^j
  - Least Squares Fit
  - Pseudo Inverse of the Scaling Matrix
    $\Delta c^j = (\Phi^j(u_0))^+ \delta$

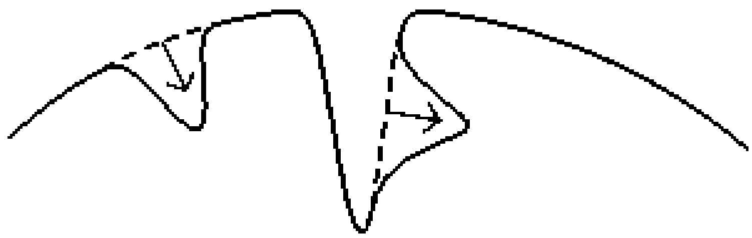
Example

- Non-Uniform Parametrization



- Changes at Fixed Scale

- Control over Area of Influence



More Control

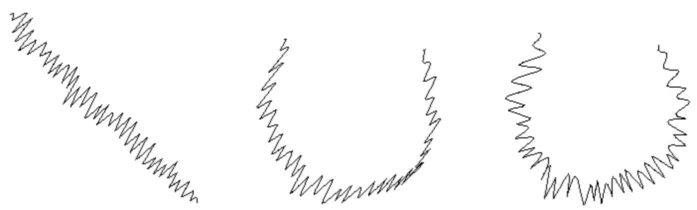
- Scale
  - Depends on Parametrization
- Intuitive Manipulation
  - Arc-Length Parametrization
  - Depends on Geometric Embedding
- Area of Influence

  h
- Computation
  - Level $l = j + t$
  - Corresponds to h at $f(u_0)$
- Estimating l

  $l = \frac{h - h^{j+}}{h^{j-} - h^{j+}}$

Orientation of Detail

- Detail
  - Displacement along the Normal
- Reference Frame
  - Tangent
  - Normal
- Options
  - Normalized
  - Non-Normalized
- Computation of Wavelet Coefficients
  - Reconstruction
  - Change of Coordinates at each Level



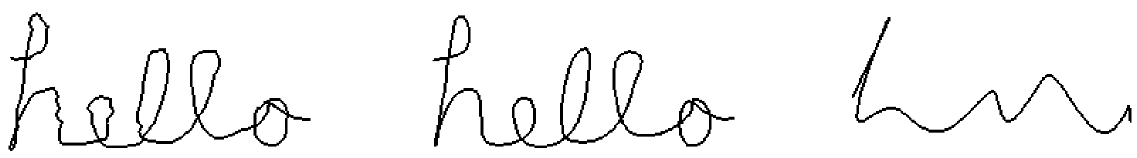
Smoothing

- Apply Decomposition
  $c^m = A^{k+1} A^{k+2} \ldots A^j c^n$

- Integer Level
  $f^m(u) = \Phi^m(u) c^m$

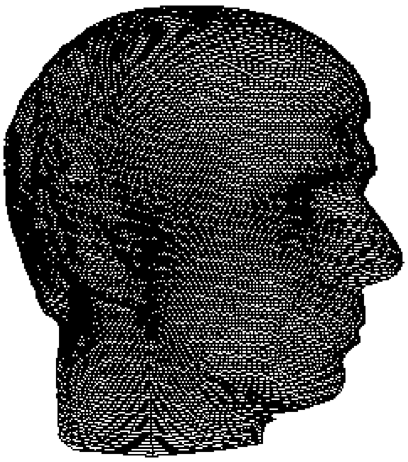
- Fractional Level
  $f^{m+t} = (1-t)\Phi^m(u)c^m + t\Phi m + 1(u)c^{m+1}$

- Example

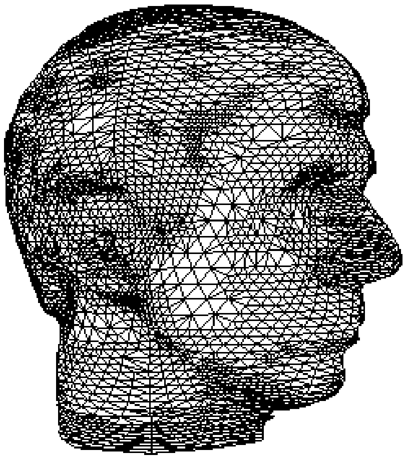


Compression

- Best Approximation
  - Original Mesh

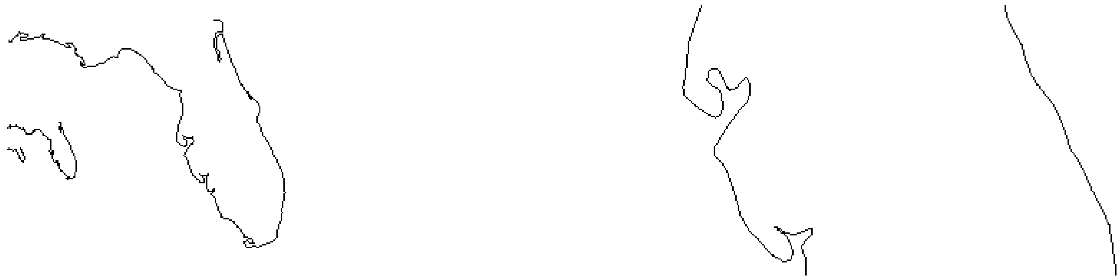


  - Decimated Mesh



Adapted Rendering

- Types
  - Resolution Compatible
  - Progressive Rendering
- Curve Display (Error Tolerance ¡ 1/400')



- Curve Models (5%, 21%, 46%, 78%)

Conclusions

- Future Research
  - Arbitrary Topology
  - Discontinuities
  - Sparse Representations
  - Model Attributes