1 Coordinates

In this chapter, we will study the systems of coordinates and the transformations for changing coordinates with emphasis in the rectilinear systems. This topic will be important further on in the study of the rotations space, in the hierarchies of articulated objects (e.g. human body), in the color spaces and in the virtual camera model.

The use of coordinates allows to obtain analytical representations of the objects and of the transformations of a geometry:

However, the representation is dependent on the coordinate system used. In this way, it is important to study coordinate change methods to change between two representations. On the other hand, as we will see, the proper choice and a coordinate system, often allow for a greater simplification of the solution.

To simplify introducing the concepts, we will begin by studying the problem in the affine $\mathbb{R}^2$-plane . (In this chapter, the points of the space $\mathbb{R}^n$ will be represented by nonbold, small or capital letters.)

1 Transforming objects, reference frames or coordinates

When scanning a book, we can position the text along the line of view in two different ways: by shifting the book or by shifting our head. This shows that two points of view exist to interpret the way a transformation works: it moves objects of the space (the book), or it changes the coordinate system (the head) leaving the objects fixed.

1.1 Transforming objects

Consider an affine transformation of the plane consisting of a rotation R by an angle θ about the origin $(0, 0)$, followed by a translation T by a vector (t_1, t_2). The total effect of this transformation is given by the product TR. For an arbitrary point $x = (x_1, x_2) \in \mathbb{R}^2$ we have $TR(x) = T(R(x))$, which means, we rotate x and then translate it.

In matrix terms we have

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

Therefore, the matrix of the composed transformation TR is given by

$$TR = \begin{pmatrix} \cos\theta & -\sin\theta & t_1 \\ \sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.2}$$

Figure 1, shows the effect of the above transformation in a unit square Q for the case of θ equals to $\frac{\pi}{4}$, and the translation vector is $(4, 2)$.

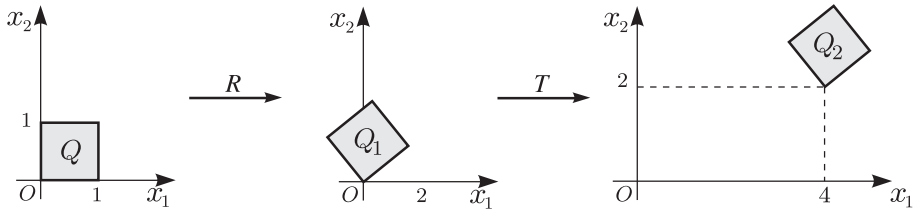

Figure 1. Transforming a square.

Notice that we just have a coordinate system involved, which is the Cartesian system. We will see another interesting example of object transformation soon afterwards.

Rotation About a Point

Our problem consists on obtaining the matrix L in Cartesian coordinates, representing the following transformation: to rotate an object by an angle θ about an arbitrary point P in the plane.

An interesting method for solving this problem, which is useful in other several circumstances, consists in expressing the wanted transformation as one product of simpler transformations. In the present case, we know that a rotation $R(x_1, x_2)$ about the origin $(0, 0)$ is given by the equation

$$R(x_1, x_2) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \tag{1.3}$$

To obtain the wanted transformation $L(y) = L(y_1, y_2)$, we translate the point y towards the origin $(0, 0)$, apply the rotation R about the origin and translate the result

to its first position. If the point P has coordinates (p_1, p_2), the translation towards the origin T_1 and its inverse T_1^{-1} are given by

$$T_1 = \begin{pmatrix} 1 & 0 & -p_1 \\ 0 & 1 & -p_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1^{-1} = \begin{pmatrix} 1 & 0 & p_1 \\ 0 & 1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

Therefore the final transformation is given by

$$TRT^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & p_1(1-\cos\theta) + p_2\sin\theta \\ \sin\theta & \cos\theta & p_2(1-\cos\theta) - p_1\sin\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

1.2 Transforming reference frames

From the chapter on Geometry we know the use of coordinates in the affine space is done by introducing a reference frame. More precisely, a coordinate system in the affine $\mathbb{R}^2$-plane is defined by a reference frame given by one point O and a basis $\{e_1, e_2\}$ of the vector space $\mathbb{R}^2$ (see Figure 2).

A point $P \in \mathbb{R}^2$ is completely determined by the vector $\overrightarrow{OP}$, and its coordinates (x_1, x_2) are defined by $\overrightarrow{OP} = x_1e_1 + x_2e_2$. In general, we referred to the coordinate system as having origin O and basis $\{e_1, e_2\}$, and we use the notationo $\mathcal{E} = (O, \{e_1, e_2\})$ or (O, e_1, e_2) for indicating it. When the origin of the system is well determined it is common to confuse the reference frame with its basis $\{e_1, e_2\}$.

The proper choice of the coordinate system can bring great simplifications to the solution of a problem. Consider, for instance, the case of describing the motion of the wheel of a car: in a coordinate system with origin in the axis of the wheel, the trajectory of each point of the wheel is a circle. On the other hand, using a coordinate system connected to the road, those trajectories would be curves of the type cycloid. However, when we use many systems of coordinates, we have the problem of describing a certain

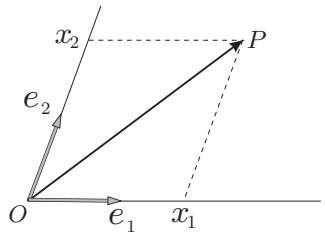

Figure 2. Coordinate system in the plane.

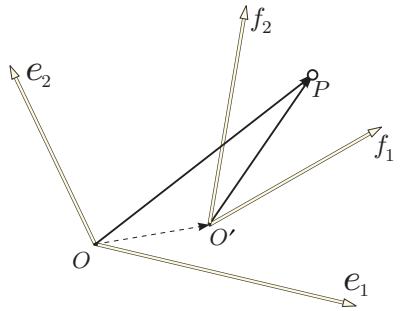

Figure 3. A point in different coordinate systems.

object in different systems. Therefore, there comes the need to obtain the transformations for changing coordinates among the several systems.

To understand transforming coordinates we will study the transformations of reference frames initially. For that, consider two reference frames $\mathcal{E} = (O, \{e_1, e_2\})$ and $\mathcal{F} = (O', \{f_1, f_2\})$ as shown in Figure 3. These reference frames can be considered objects of the geometry, therefore it makes sense to look for a transformation A that takes the reference frame $\mathcal{E}$ into the reference frame $\mathcal{F}$. In a similar way, to the that we saw in the previous section, this transformation is obtained in two stages:

- We determine the linear transformation L that takes the basis $\{e_1, e_2\}$ into the basis $\{f_1, f_2\}$;
- We determine the translation T that takes the origin O of the reference frame $\mathcal{E}$ into the origin O' of the reference frame $\mathcal{F}$.

The transformation A is then obtained by making the product $A = TL$.

Supposing the coordinates of the vector $\overrightarrow{OO'}$ in basis $\{e_1, e_2\}$ be

$$\overrightarrow{OO'} = t_1 e_1 + t_2 e_2, \tag{1.5}$$

matrix of T will be given by

$$T = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

Besides, if

$$\begin{aligned} f_1 &= L(e_1) = a_{11} e_1 + a_{21} e_2; \\ f_2 &= L(e_2) = a_{12} e_1 + a_{22} e_2, \end{aligned} \tag{1.7}$$

, the transformation L is given by the matrix

$$L = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.8}$$

That is the *transfer matrix* of the basis $\{e_1, e_2\}$ for the basis $\{f_1, f_2\}$.

Using the matrices of T and L in (1.6) and (1.8), we can write:

$$A = TL = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{11} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.9}$$

That is, the *transfer matrix* of the reference frame $\mathcal{E}$ for the reference frame $\mathcal{F}$. We will use the notation $A_{\mathcal{E}}^{\mathcal{F}}$ when there is the need to specify the reference frames. Notice that the first column of that matrix supplies the coordinates of the vector $f_1 = L(e_1)$ in the reference frame system $\{e_1, e_2\}$, the second column supplies the coordinates of the vector $f_2 = L(e_2)$, and the third column supplies the coordinates of the origin $O' = L(O)$ of the new system in relation to the original reference frame system $(O, \{e_1, e_2\})$.

1.3 Transforming coordinates

Here we do not have innovations: we only used the coordinate system of the reference frame $\mathcal{E}$, and the elements of the reference frame $\mathcal{F}$, origin and vectors, all expressed in relation to that system. The relevant question now is:

> "*How does the transfer matrix $A_{\mathcal{E}}^{\mathcal{F}}$ relate the coordinates in the system of the reference frames $\mathcal{E}$ and $\mathcal{F}$?*"

To answer that question, we will relate the coordinates of an arbitrary point P of the plane in the systems of coordinates of the reference frame $\mathcal{E}$ and in the system of the reference frame $\mathcal{F}$ (see Figure 3). In the system $\mathcal{E}$ the point P is defined by the vector $\overrightarrow{OP}$, and in the system $\mathcal{F}$, P is defined by the vector $\overrightarrow{O'P}$. We immediately obtain the vector equation relating P in the two systems:

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}. \tag{1.10}$$

To obtain the relation of the coordinates of P in the two systems, we will write the above vector equation using coordinates.

Let us indicate by (y_1, y_2) the coordinates of P in the system $(O', \{f_1, f_2\})$, and by (x_1, x_2) the coordinates of P in the system $(O, \{e_1, e_2\})$. That is,

$$\begin{aligned} \overrightarrow{OP} &= x_1 e_1 + x_2 e_2 \\ \overrightarrow{O'P} &= y_1 f_1 + y_2 f_2 \end{aligned} \tag{1.11}$$

Replacing the expressions of $\overrightarrow{OP}, \overrightarrow{O'P}$ in (1.11), and of $\overrightarrow{OO'}$ in (1.5), in equation (1.10), we get

$$x_1e_1 + x_2e_2 = t_1e_1 + t_2e_2 + y_1f_1 + y_2f_2.$$

Taking into account the values of f_1 and f_2 in equation (1.7), we have the following for the above equation

$$\begin{aligned} x_1e_1 + x_2e_2 &= t_1e_1 + t_2e_2 + y_1(a_{11}e_1 + a_{21}e_2) + y_2(a_{12}e_1 + a_{22}e_2) \\ &= (t_1 + y_1a_{11} + y_2a_{12})e_1 + (t_2 + y_1a_{21} + y_2a_{22})e_2. \end{aligned}$$

We therefore conclude that the change of coordinates (y_1, y_2) of reference frame $\mathcal{F}$ to the coordinates (x_1, x_2) of reference frame $\mathcal{E}$ is given by

$$\begin{aligned} x_1 &= t_1 + y_1a_{11} + y_2a_{12} \\ x_2 &= t_2 + y_1a_{21} + y_2a_{22}, \end{aligned}$$

or, using matrices,

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}. \tag{1.12}$$

Notice that this is exactly the transfer matrix $A_{\mathcal{E}}^{\mathcal{F}}$ of the reference frame $\mathcal{E}$ for the reference frame $\mathcal{F}$, obtained previously in the equation (1.9). We will highlight the above fact above which brings a lot of confusion:

> "*The transfer matrix $A_{\mathcal{E}}^{\mathcal{F}}$ changes the coordinates of a point in the system $\mathcal{F}$ for the coordinates in the system $\mathcal{W}$.*"

In other words, the coordinates change in a way contrary to the reference frame.

Example 1. Be $\mathcal{E} = (O, \{e_1, e_2\})$, where $O = (0, 0)$, $\{e_1, e_2\}$ is a canonical basis of $\mathbb{R}^2$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Be $\mathcal{F}$ the system with origin at point $O' = (4, 2)$ in which the basis vectors f_1 and f_2 are obtained from the vectors e_1 and e_2 by a rotation of $\pi/4$ (45° degrees) in the counter-clockwise direction (see Figure 4(a)). In this case we have

$$\begin{aligned} f_1 &= \cos\frac{\pi}{4}e_1 + \sin\frac{\pi}{4}e_2; \\ f_2 &= -\sin\frac{\pi}{4}e_1 + \cos\frac{\pi}{4}e_2; \\ O' &= 4e_1 + 2e_2. \end{aligned}$$

Therefore the matrix of the transformation T changing the reference frame $(O, \{e_1, e_2\})$ for the reference frame $(O, \{f_1, f_2\})$ is given by the product

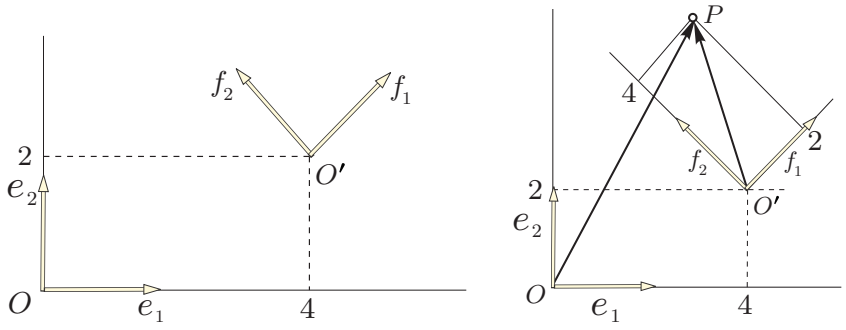

Figure 4. References (a); change of coordinates (b).

$$T = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 4 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.13}$$

Now be P the point of coordinates $(2, 4)$ in relation to the system $\mathcal{F}$. Applying the matrix (1.13) we obtain the point $(4-\sqrt{2}, 2+3\sqrt{2})$ that represents the coordinates of P in the reference frame system $\mathcal{E}$. Figure 4 (b) shows the vectors $\overrightarrow{OP}$ and $\overrightarrow{O'P}$ representing the point P in each one of the systems. ◁

Inverse Transformation

Previously, we saw how to obtain the affine transformation $L_{\mathcal{E}}^{\mathcal{F}}$ that makes the transfer of the reference frame $\mathcal{E}$ for the reference frame $\mathcal{F}$, changing the coordinates of the system $\mathcal{F}$ for the coordinates in $\mathcal{E}$. It is easy to see the transformation that changes the reference frame $\mathcal{F}$ for the reference frame $\mathcal{E}$ (and therefore changing the coordinates of $\mathcal{E}$ for the coordinates of $\mathcal{F}$), is given by inverse L^{-1} of the transformation L. That is, $L_{\mathcal{F}}^{\mathcal{E}} = (L_{\mathcal{E}}^{\mathcal{F}})^{-1}$.

Therefore, if (x_1, x_2) are the coordinates in the reference frame $(O, \{e_1, e_2\})$ and (y_1, y_2) are the coordinates in the reference frame $(O', \{f_1, f_2\})$, a transformation T changes the coordinates (y_1, y_2) for the coordinates (x_1, x_2). An inverse transformation T^{-1} changes the coordinates (x_1, x_2) for (y_1, y_2).

2 Local and global transformations

In applications, in general, we have a global coordinate system referencing the whole scene. In Computer Graphics such system is called *world coordinate system*. We also use

specific systems of coordinates in some objects of the world because, in their own system, the equations involving the coordinates are simpler and this is reflected in formulating the problems. Those systems receive the name of *local coordinate systems*.

A typical case happens when we want to describe n stages of the motion of a rigid body in the Euclidean space. As the body is rigid, each stage is described by a position and an orientation. We also associate a local reference frame to the rigid body. By doing this, each stage of the motion is specified by a change of coordinates from the reference frame of the local space to the global. To study this type of problem it is important to set up an equation for the consecutive change among reference frames.

Consider three reference frames $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, and suppose we know the transformations for changing from $\mathcal{E}_1$ to $\mathcal{E}_2$, and from $\mathcal{E}_2$, to $\mathcal{E}_3$. How to find this transformation changing from reference frame $\mathcal{E}_1$ to reference frame $\mathcal{E}_3$?

> "*the transformation $L_{\mathcal{E}_1}^{\mathcal{E}_3}$, changing from reference frame $\mathcal{E}_1$ to reference frame $\mathcal{E}_3$ is obtained by the product $L_{\mathcal{E}_1}^{\mathcal{E}_3} = L_{\mathcal{E}_1}^{\mathcal{E}_2} L_{\mathcal{E}_2}^{\mathcal{E}_3}$.*"

Demonstrating this fact is simple: given a point $z = (z_1, z_2)$ in the reference frame $\mathcal{E}_3$, the transformation $L_{\mathcal{E}_2}^{\mathcal{E}_3}(z_1, z_2)$ obtains the coordinates of z in the reference frame $\mathcal{E}_2$, and therefore, the transformation $L_{\mathcal{E}_1}^{\mathcal{E}_2}(L_{\mathcal{E}_2}^{\mathcal{E}_3}(z_1, z_2))$ supplies the coordinates of z in the system of the reference frame $\mathcal{E}_1$. This means the transformation $L_{\mathcal{E}_1}^{\mathcal{E}_2} L_{\mathcal{E}_2}^{\mathcal{E}_3}$ changes the coordinates of z for the initial system $\mathcal{E}_1$. The above result is very useful in applications. We will discuss it soon afterwards with more details.

Suppose that we have an initial reference frame (global reference frame) given by

$$\mathcal{E}^0 = (O^0, \{b_1^0, b_2^0, \ldots, b_n^0\}),$$

and n successive (local) reference frames indicated by

$$\begin{aligned}
\mathcal{E}^1 &= (O^1, \{b_1^1, b_2^1, \ldots, b_n^1\});\\
\mathcal{E}^2 &= (O^2, \{b_1^2, b_2^2, \ldots, b_n^2\});\\
\mathcal{E}^3 &= (O^3, \{b_1^3, b_3^3, \ldots, b_n^3\});\\
&\vdots\\
\mathcal{E}^n &= (O^n, \{b_1^n, b_n^n, \ldots, b_n^n\}).
\end{aligned}$$

We will indicate as T_{i-1}^i the transformation that takes the reference frame $\mathcal{E}^{i-1}$ to the reference frame $\mathcal{E}^i$, that is, T_{i-1}^i is a local transformation specified in the reference frame $\mathcal{E}^{i-1}$. The product

$$T = T_0^1 T_1^2 T_2^3 \cdots T_{n-1}^n \tag{1.14}$$

of the local transformations, in that order, takes the initial reference frame $\mathcal{E}^0$ in the reference frame $\mathcal{E}^n$, and therefore changes the coordinates of the local reference frame $\mathcal{E}^n$ for the global reference frame $\mathcal{E}^0$.

The order of the product in (1.14) is very convenient for Computer Graphics:

- It is easier for the user to specify the transformations T_{i-1}^{i} once they are local transformations;
- The implementation is easily obtained with stack structures, because the transformations are specified in the order $T_0^1 \to T_1^2 \to \cdots \to T_{n-1}^n$ and they are applied in the inverse order $T_{n-1}^n \to T_{n-2}^{n-1} \to \cdots \to T_2^1 \to T_1^0$.

The above result is very useful to obtain change of coordinates when we have several reference frames in a hierarchy[1], where the local transformations are easy of be obtained: we obtain the sequence of local transformations and we multiply in the order that we saw above. Let us see the example of a planar mechanical arm (one can think as a two-dimensional robot's arm).

Example 2. Consider the two-dimensional mechanical arm shown in Figure 5 (the). The forearm is fastened to the wall, and it has length d_1. The arm has length d_2 and it rotates about the forearm. The hand has length d_3 and it rotates about the arm. Problem: *given a point P at the end of the hand, obtain the coordinates of P in the plane when the hand is moved.*

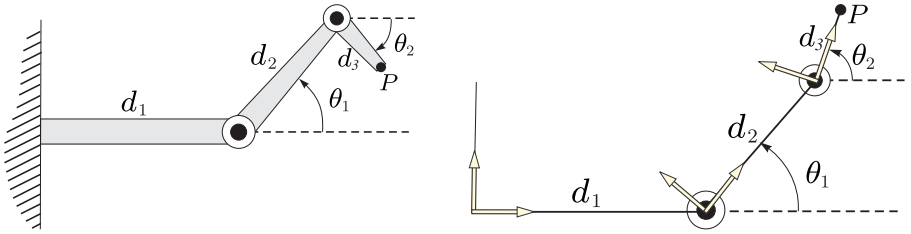

Figure 5. Global transformations (a); local transformations (b).

Of course, we can solve the problem as one simple exercise of trigonometry (we left this task to the reader). We will solve the problem using changes of successive coordinates. We have three reference frames (Figure 5 (b)): The global system of the forearm, that is fastened to the wall, the system of the arm, and the system of the hand.

1. We take as a global system $\mathcal{E}$, the system of the forearm (that is fastened). That is the Cartesian system of the plane whose origin O is the intersection of the forearm with the wall and whose basis is formed by the canonical vectors of the $\mathbb{R}^2$, e_1 and e_2.

[1] we will do a detailed study of hierarchies in a chapter further on.

2. The coordinate system of the arm, which is obtained from the global system by a translation of d_1 in the direction of e_1 and by positioning the reference frame in a way that vector f_1 follows the rotation of the arm about the connection joint with the forearm.

3. Similarly, the system of the hand is obtained from the system of the forearm making a translation from the origin along vector f_1 of length d_2 and one rotation by an angle θ_2.

Notice that we have two consecutive transformations of reference frames T_1 and T_2 from the global reference frame to the reference frame of the hand:

$$\text{Forearm} \xrightarrow{T_1} \text{Arm} \xrightarrow{T_2} \text{Hand}.$$

Therefore, the transformation from the global reference of the forearm to the system of the hand is given by $L = T_1T_2$.

The transformation T_1 is a translation of d_1 along the vector e_1, followed by one rotation by an angle θ_1. The transformation T_2 is a translation of d_2 along the vector f_1, followed by one rotation by an angle θ_2. In terms of matrices we have:

$$T_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & d_1 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & d_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

The final transformation is then given by

$$L = T_1T_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & d_1 + d_2\sin\theta_1 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & d_2\cos\theta_1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.16}$$

We now observe that, in the system of the hand, the point P has coordinates $(d_3, 0)$, therefore, to determine the coordinates of the position of the point P, it is enough to apply the above matrix in the vector $(d_3, 0, 1)$. ◁

3 Rectilinear coordinates of space

Everything studied in the previous section about systems of coordinates of the plane, is generalized naturally for $\mathbb{R}^n$. In general, one coordinate system is defined by one basis $\beta = \{e_1, \ldots, e_n\}$ and one point $O \in \mathbb{R}^n$. The pair (O, β) is called *reference frame of the space*, and the point O is called origin of the system. The coordinates of a point $P \in \mathbb{R}^n$ in that system are given by the components of the vector $\overrightarrow{OP}$ in the basis β. If

$$\overrightarrow{OP} = \sum_{j=1}^{n} x_j e_j,$$

we write $P = (x_1, \ldots, x_n)$, being clear which reference frame is on use.

Given another reference frame (O', γ), where γ is the basis $\gamma = \{f_1, \ldots, f_n\}$, we define a linear transformation $R\colon \mathbb{R}^n \to \mathbb{R}^n$, that takes the basis $\{e_1, \ldots, e_n\}$ into the basis $\{f_1, \ldots, f_n\}$, that is, $R(e_i) = f_i$. This transformation is invertible, and its inverse takes the basis γ in the basis β.

The translation T of $\mathbb{R}^n$ given by the vector $\overrightarrow{OO'}$ takes the origin of the reference frame (O, β) into the origin of the reference frame (O', γ) (see Figure 6).

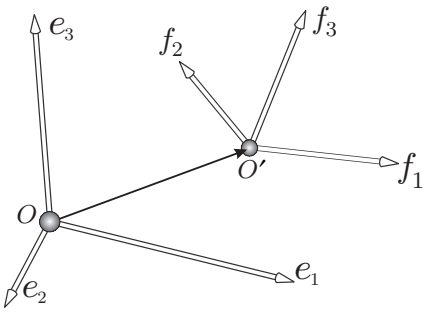

Figure 6. Reference frames in space.

The composed $TR\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an affine transformation that takes the reference frame (O, β) into the reference frame (O', γ). Using homogeneous coordinates, the matrix $A = (a_{ij})$ of TR is obtained in the following way: for $j = 1, 2, \ldots, n$, the jth column of A is given by

$$(a_{1j}\ a_{2j}\ a_{3j} \ldots a_{nj}\ 0),$$

where a_{ij} are the coordinates of vector f_j in basis $\{e_1, \ldots, e_n\}$, that is,

$$f_j = \sum_{i=1}^{n} a_{ij} e_i.$$

The last column of the matrix A is formed by the coordinates of the translation vector $\overrightarrow{OO'}$ in the basis $\{e_1, \ldots, e_n\}$.

Given $Y = (y_1, \ldots, y_n)$ as the coordinates of a point P in the system $F = (O', \{f_1, \ldots, f_n\})$, and $X = (x_1, \ldots, x_n)$ as coordinates in the reference frame system $E = (O, \{e_1, \ldots, e_n\})$. The transformation for changing coordinates is therefore given by

$$X = TRY.$$

When $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are orthonormal bases, a transformation TR for changing systems is a rigid motion of the space: R i an orthogonal transformation, and T is a translation. In that case, we say that the systems of coordinates are *orthogonal*.

Example 3. Consider the system $\mathcal{E} = (O, \{e_1, e_2, e_3\})$ of Cartesian coordinates of $\mathbb{R}^3$ and the orthonormal reference frame $\mathcal{F} = (O', \{u, v, n\})$, where $O' = (o_x, o_y, o_z)$, $u = (u_x, u_y, u_z)$, $v = (v_x, v_y, v_z)$ e $n = (n_x, n_y, n_z)$, as illustrated in Figure 7.

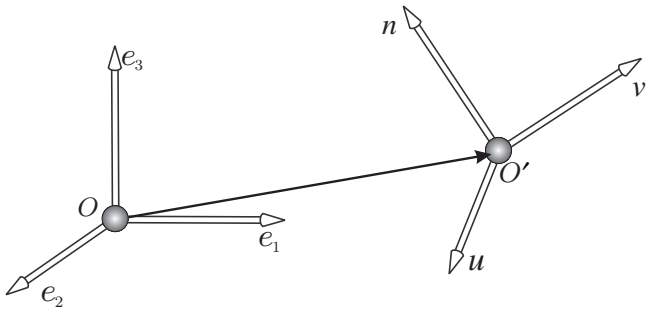

Figure 7. Orthonormal reference frames in space.

The transformation T from reference frame $\mathcal{E}$ to reference frame $\mathcal{F}$ is defined by the matrix

$$T = \begin{pmatrix} u_x & v_x & n_x & o_x \\ u_y & v_y & n_y & o_y \\ u_z & v_z & n_z & o_z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transformation that takes the coordinates Cartesian (x, y, z) of a point in $\mathbb{R}^3$ into the coordinates (u, v, n) in the defined system for the reference frame $\mathcal{F}$ is given by the inverse of T. A simple calculation, taking into account that matrix T is orthogonal, shows that matrix

$$\begin{pmatrix} u_x & u_y & u_z & -\langle \overrightarrow{OO'}, u \rangle \\ v_x & v_y & v_z & -\langle \overrightarrow{OO'}, v \rangle \\ n_x & n_y & n_z & -\langle \overrightarrow{OO'}, n \rangle \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

is the inverse of matrix T. ◁

3.1 An application: Position of a robot's arm in space

Consider a simplified model of a robot's arm shown in Figure 8 (a). This model consists of an articulated set of two rigid parts corresponding to the arm and the forearm. The connection of the arm with the forearm have a joint that allows a rotation (notice that both arm and forearm are in the same plane). Besides, the forearm (and therefore the arm) can rotate about an axis.

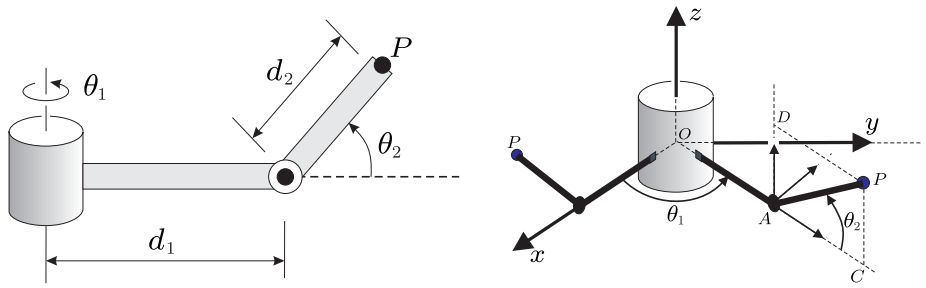

Figure 8. Arm's hierarchy (a) and coordinate system (b).

Our problem consists on determining the coordinates of the point P in the Cartesian system of the space, when the arm is moved. We will use two methods to solve the problem: the direct calculation, and the calculation using local transformations.

Direct calculation of coordinates

Figure 8 (b) shows the arm in two different positions after rotating the forearm and the arm by angles θ_1 and θ_2, respectively. Notice that the trajectories of the point P in the extremity of the arm are contained in a torus of rays d_1 and d_2. By observing that figure, we immediately obtain the coordinates of P, as follows,

$$\begin{aligned} x &= OC\cos\theta_1 = (OA + AC)\cos\theta_1; \\ y &= OC\sin\theta_1 = (OA + AC)\sin\theta_1; \\ z &= OB\sin(-\theta_2). \end{aligned}$$

Notice that the sign in the angle θ_2 is due to the fact that the rotation is done in the counter-clockwise direction in relation to the orientation of the z and y axes.

Taking into account that $OA = d_1$ and $AC = AP\cos\theta_2 = d_2\cos\theta_2$, we obtain the coordinates of point P:

$$\begin{aligned} x &= (d_1 + d_2\cos\theta_2)\cos\theta_1 \\ y &= (d_1 + d_2\cos\theta_2)\sin\theta_1 \\ z &= d_2\sin(-\theta_2). \end{aligned}$$

As expected, those are the parametric equations of a torus of rays d_1 and d_2.

Calculation using local transformations

Let us now solve the problem using successive local transformations. A transformation of the Cartesian system for the reference frame system of the upperpart of the arm is given by the rotation by an angle θ about the z - axis.

$$A_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To obtain the reference frame of the lower part of the arm, starting from the reference frame of the upper arm, we make a translation by the vector $(d_1, 0, 0)$, following by a rotation by an angle $-\theta_2$ about the y - axis. The matrix is given by

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\theta_2) & 0 & -\sin(-\theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\theta_2) & 0 & \cos(-\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the coordinate system for the hand is obtained starting from the system of the lower arm performing one translation of length d_2 in the direction of the x - axis. The matrix is given by

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The final matrix is obtained by the product $T = A_1 A_2 A_3$. One simple calculation shows that

$$T = \begin{pmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1 & \cos\theta_1\sin\theta_2 & (d_2\cos\theta_1 + d_1)\cos\theta_1 \\ \sin\theta_1\cos\theta_2 & -\cos\theta_1 & \sin\theta_1\sin\theta_2 & (d_2\cos\theta_2 + d_1)\sin\theta_1 \\ -\sin\theta_2 & 0 & \cos\theta_2 & -r\sin\theta_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice now that point P of the hand is the origin of the coordinate system of the hand. Therefore, that point has homogeneous coordinates $(0, 0, 0, 1)$. Applying matrix T above, we obtain the coordinates of the point in the global system of the space:

$$\begin{aligned} x &= (d_2\cos\theta_1 + d_1)\cos\theta_1; \\ y &= (d_2\cos\theta_2 + d_1)\sin\theta_1; \\ z &= -r\sin\theta_2, \end{aligned}$$

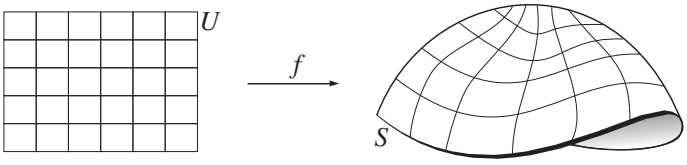

Figure 9. Curvilinear coordinates.

as expected, that is the same value obtained previously using the direct method.

AT first sight, the use of the local transformations method to calculate positioning of point P seems more complicated than the direct method described in the previous section. However, several reasons favor the local transformations method. Among them, we can highlight the fact the method is general and it can be used for an arm with a larger number of articulations (or even for objects with more complicated hierarchies). In reality, the direct method becomes more difficult of being used for an arm with three movable parts (forearm, arm, hand). We will return to this subject further on in the chapter on hierarchies.

4 Coordinates and parameterization

Given a space S and a subset $U \subset \mathbb{R}^n$, a *coordinate system* in S is an application $f\colon U \subset \mathbb{R}^n \to S$ that is bijective with class C^1, at least. The idea of the coordinate system is to allow working with the space S starting from known of the space $\mathbb{R}^n$, by mapping the concepts using the application f (see Figure 9).

The bijective of f guarantees us that a point $p \in S$ is represented by the point $f^{-1}(p) \in \mathbb{R}^n$; if $f^{-1}(p) = (x_1, \ldots, x_n)$, we say that $(x_1, \ldots, x_n)$ are the coordinates of point p. The curves $c_i(t) = f(x_1, \ldots, t, \ldots, x_n)$, $t \in \mathbb{R}$, are called *coordinate curves* of the space S.

The differentiability of f guarantees that concepts involving differential calculus (e.g., derivatives, differential equations) can be too defined in S by using coordinates. (In reality, to avoid problems, we should also have the inverse of f at least of class C^1; In addition, the derivative of f should be nonull in all points).

In general, we intrinsically define the concepts in space S (that is without depending on coordinates) and we use coordinates when we need to do calculations in that space. We can have an infinity of systems of coordinates in a space. Previously, we studied several rectilinear systems of coordinated in the space $\mathbb{R}^n$. Those systems are called rectilinear because the coordinate curves are straight. The systems of coordinates that do not hold that property are called *curvilinear* (see Figure 9).

An important problem is the study of curvilinear systems of coordinates, in particular

defining such systems in spaces that do not hold a linear structure. Defining one coordinate system in a space S is also known as *space parameterization*. Unfortunately, because of book's space constraints, we will not be giving a proper coverage to that topic, despite its importance in Computer Graphics.

Below, we will present a parameterization example to highlight the problems involved.

4.1 Spherical coordinates

The unit sphere of $S^2 \subset \mathbb{R}^3$,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\},$$

holds a natural parameterization known as *spherical coordinates*, based on the fact the fact that each point of S^2 is characterized by the longitude and latitude angles θ and ϕ, respectively, as shown in Figure 10. (It is also common to use the azimuth angle ψ instead of the latitude angle: $\psi = \pi/2 - \phi$.)

From figure 10 we have

$$\begin{aligned} x &= x(\theta, \phi) = \cos\phi\cos\theta; \\ y &= y(\theta, \phi) = \cos\phi\sin\theta; \\ z &= z(\theta, \phi) = \sin\phi. \end{aligned}$$

Therefore, the application $f : U \subset \mathbb{R}^2 \to S^2$, defined by

$$f(\theta, \phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi),$$

defines a coordinate system in the sphere. Notice that for having bijectivity we should restrict the domain U to the open subset $(0, 2\pi)\times(-\pi/2, \pi/2)$ in the plane (see Figure 10).

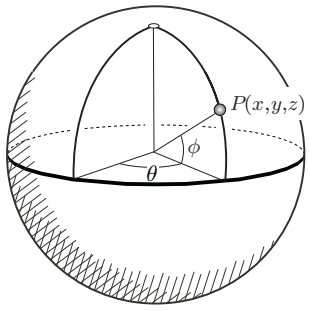

Figure 10. Spherical coordinates.

Restricting the parameterization to the domain $(0, 2\pi) \times (-\pi/2, \pi/2)$ results in our coordinate system not covering the whole sphere. On the other hand, if we do not restrict the parameterization domain, for $\phi = \pi/2$ we have $f(t, \pi/2) = (0, 0, 1)$, that is, the straight line $\mathbb{R} \times \pi/2$ on the plane is mapped to a single point (the north pole of the sphere).

In the above case we say the parameterization introduces a *singularity*. In singularities, the derivative of the function defining the coordinate system does not have maximum rank. Therefore, the system is not adapted to treat concepts that use differentiability.

The duality seen above is a certain problem when defining systems of coordinates in spaces that do not hold a linear structure: either we build systems with singularities or ones without covering the whole space. The latter are called *systems of local coordinates*, and they just define a *local parameterization*. We will have the opportunity to cover this problem further ahead when parameterizing the rotations space of $\mathbb{R}^3$,

5 Comments and references

Changing among rectilinear systems of coordinates is a core and basic topic of Linear ALgebra and Affine Geometry. However, students might lack familiarity with various nuances present in the problems. For this reason we decided to append this chapter to the book.

For readers interested on revising the concepts of Linear Algebra, we recommend (**?**).

5.1 Additional topics

We just superficially approached the subject of curvilinear coordinates. This study is immediately reduced to the study of differentiable applications of $\mathbb{R}^n$, a topic that is studied in calculus of several variables, and it is consolidated in the study of Analysis in the $\mathbb{R}^n$. In particular, it is important to discuss and understand the problem of changing of coordinates.

That subject is directly related to the study of nonrigid space deformations that finds countless applications in the Computing Graphics area of Warping & Morphing.

The topic of local coordinates and hierarchies, covered with some examples in this chapter, will be approached in more depth in the chapter of hierarchies further on.

6 Exercises

1. Consider the reference frame of the affine $\mathbb{R}^2$-plane, $\mathcal{F} = (O, \{f_1, f_2\})$ where $O = (3, -2)$, $f_1 = e_1 + 3e_2$ and $f_2 = -2e_1 + 4e_2$.

a) Find the matrix fro changing from the Cartesian reference frame the reference frame $\mathcal{F}$;

b) Find the coordinates of a point $P = (x_1, x_2)$ of the Cartesian system in the system of the reference frame $\mathcal{F}$.

2. Consider the reference frame $\mathcal{F}$ in $\mathbb{R}^2$ obtained from the Cartesian reference frame rotating 135° about the origin in the counter-clockwise direction. Rewrite the equation $5x^2 + 6xy + 5y^2 = 1$, where (x, y) are Cartesian coordinates, in coordinates (s, t) of the reference frame $\mathcal{F}$.

3. Consider the quadratic form $\varphi(x, y, z) = 7x^2+6y^2+5z^2-4xy-4yz$, where (x, y, z) are Cartesian coordinates of $\mathbb{R}^3$. Find the equation of φ in the coordinate system defined by the reference frame $\mathcal{F} = ((0, 0, 0), \{f_1, f_2, f_3\})$ where $f_1 = \frac{1}{3}(-2e_1 + 2e_2 + e_3)$, $f_2 = \frac{1}{3}(2e_1 + e_2 + -2e_3)$ and $f_3 = \frac{1}{3}(e_1 + 2e_2 + -2e_3)$.

4. Consider the planar curve with equation $x^2 + y^2 + 2x - 3y + 9 = 0$. Obtain the equation of this curve in the coordinate system with origin $(2, 3)$ and whose basis is the canonical basis of $\mathbb{R}^2$.

5. Consider the equation of the second degree $5x^2 + 9y^2 + 6xy = 8$. Write this equation in the coordinate system whose reference frame has origin $(0, 0)$, and whose basis is obtained from the canonical basis of $\mathbb{R}^2$ by making one rotation by $\pi/4$ in the counter-clockwise direction

6. Consider a straight line r in the $\mathbb{R}^2$-plane. Define a shear transformation along r and centered at a point $O \in r$ (see Figure 11). Find the matrix of this transformation in the Cartesian coordinate system.

7. Repeat the previous exercise considering a straight line of the $\mathbb{R}^3$-space, defined by one unit vector $u = (u_1, u_2, u_3)$.

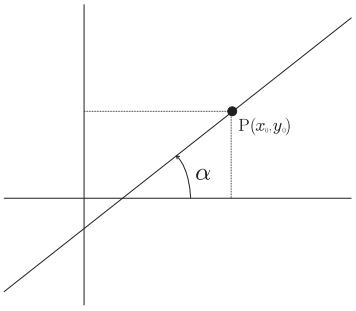

Figure 11. Ratio between segments.

8. Consider a rotation by an angle θ about one defined axis for the unit vector $r = (r_x, r_y, r_z)$. Show that the transformation matrix is given by

$$\begin{pmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)+r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{pmatrix},$$

(Suggestion: use the steps of the method we used in this chapter to find a rotation of the plane about an arbitrary point.)

9. Find the transformation matrix for a rotation by a 120° angle about the axis defined by the unit vector $r = \frac{1}{\sqrt{3}}(1,1,1)$.

10. Let

$$T = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be the matrix of a rigid motion of the space in homogeneous coordinates. Show that the inverse transformation is given by

$$T^{-1} = \begin{pmatrix} n_x & n_y & n_z & -\langle p, n\rangle \\ o_x & o_y & o_z & -\langle p, o\rangle \\ a_x & a_y & a_z & -\langle p, a\rangle \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $o = (o_x, o_y, o_z)$, $a = (a_x, a_y, a_z)$, $n = (n_x, n_y, n_z)$, and $p = (p_x, p_y, p_z)$. (Suggestion: the first three columns of the matrix are made of orthonormal vectors.)

11. Formulate the problem of scale changing in a coordinate system of the $\mathbb{R}^n$ in terms of changing the coordinate system, and of the matrix of changing the systems.

12. Suppose the cylindrical basis of the mechanical arm described in Section 3.1 moves along the diagonal of the plane $z = 0$ in the space. Which change should be made in matrix T that parameterizes the positioning of the arm in the space?

13. Consider the mechanical arm shown in Figure 12, composed of three parts: forearm, arm and hand. Similarly to what we did in the Section 3.1, find the positioning coordinates of a point P in the hand of the arm. (Suggestion:: Use the local transformations method. It is not necessary to compute the product of the matrices.)

14. Show that a rotation of a point $p = (x, y)$ in the plane by an angle θ about the origin can be obtained by the product $e^{i\theta}p$ of the complex number $p = x + iy$ for the unit compound $e^{i\theta} = \cos\theta + i\sin\theta$.

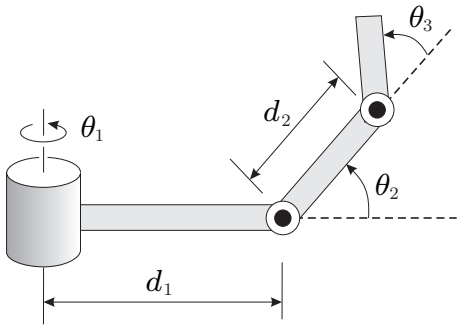

Figure 12. Hierarchy of an arm.

15. Using the result from the previous exercise, let $z_1 = e^{i\theta_1}$ and $z_2 =^{i\theta_2}$ be two rotations in the plane. Consider the following three expressions to obtain a rotation $R(t)$ by interpolating those rotations:

1. $R_1(t) = (1-t)z_1 + tz_2$;
2. $R_2(t) = \dfrac{(1-t)z_1 + tz_2}{|(1-t)z_1 + tz_2|}$;
3. $R_3(t) = e^{(1-t)i\theta_1 + ti\theta_2}$.

Discuss the above three expressions by analyzing the advantages and disadvantages of each one of them. Next, show that $R_3(t) = (z_2 z_1^{-1})^t z_1$.

16. Consider the parameterization of the sphere S^2 by spherical coordinates discussed in the text. Find the image, $(O', \{f_1, f_2\})$, of the canonical reference frame $(O, \{e_1, e_2\})$ in the sphere.

17. Find a parameterization of the cylinder $S : x^2 + y^2 = 1$ of $\mathbb{R}^3$ for a function $f\colon \mathbb{R}^2 \to S$ defined by $f(\theta, t) = (\cos\theta, \sin\theta, t)$.

a) Show this parameterization does not hold singularities.

b) Define a maximum domain of f in which it defines one coordinate system in the cylinder (*cylindrical coordinates*).

c) Find the image of the canonical reference frame $(O, \{e_1, e_2\})$ for that parameterization.

Bibliography

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Índice