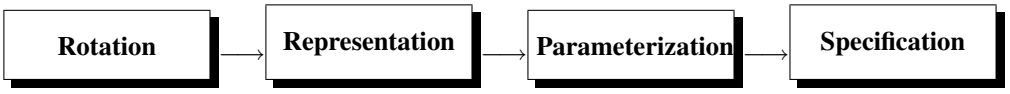
4 Rotations

The space of rotations has great importance in Computer Graphics. Examples of applications include animation of rigid bodies, linked hierarchies, as well as the specification of external parameters for orientation and position of the virtual camera:

In this chapter, we are interested on two problems:

> "*to obtain good representations and parameterizations of the space of rotations.*"

In general, a good representation leads to a good parameterization. The latter, in turn, leads to good interface solutions for the user to specify and to manipulate rotations in the computer:



What is a good representation or a good parameterization as shown above? Current literature presents six methodologies to describe and to specify rotations in Computer Graphics: Interpolation (e.g. keyframes in animation); Direct and inverse kinematics; Direct and inverse dynamics and Space-time optimization.

A good representation of the space of rotations should attend the requirements of these several methodologies. Among them we can mention:

1. Capability of determining orientations of rigid bodies in space as a function of the parameters;

2. The possibility to calculate the parameters from a rotation;

3. The possibility to calculate the physical magnitudes of Kinematics and of the Dynamics associated with a rigid body.

4. In particular, the previous item demands the possibility to (1) do calculations derived from the positions and orientations as a function of the parameters, (2) to obtain equations of several types (differential, integral, etc.) and (3) to solve those equations.

5. Possibility to have a good control in the interpolation of rotations, by means of the interpolation of the associated parameters;

6. The possibility to do operations among rotations either in the parameters or the representation space.

We begin our study with a brief discussion about rotations on the plane.

1 Rotations in the Euclidean Plane

The space of rotations in the plane is indicated by $\mathrm{SO}(2)$ and it is called two-dimensional *special orthogonal group*. Given an orthonormal basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ of the plane, the rotation R that carries the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to basis β is defined by the angle θ between vectors $\mathbf{e}_1$ and $\mathbf{b}_1$, as shown in Figure 1(a). The matrix of R is given by

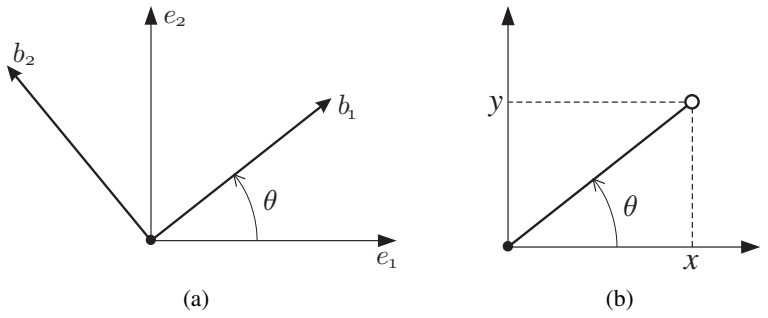
$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.1}$$

Therefore, we have a natural parameterization of $\mathbb{R}\colon R \to \mathrm{SO}(2)$, $R = R(\theta)$. However, the parameterization is not injective because $R(\theta) = R(\theta + 2k\pi)$. We can have bijectivity by restricting the domain to the interval $[0, 2\pi)$.

Instead of using matrices, we can represent the space $\mathrm{SO}(2)$ using complex numbers. In fact, given an arbitrary point $\mathbf{p} = (x, y)$ of the plane, we identify $\mathbf{p}$ with the complex number $x + iy$. We can then write $\mathbf{p}$ in its polar form,

$$\mathbf{p} = re^{i\theta_p} = r\cos\theta_p + ri\sin\theta_p,$$

where $r = \sqrt{x^2 + y^2}$ is the *norm* of $\mathbf{p}$, and $\theta_p = \mathrm{arctg}(y/x)$ is the angle from $\mathbf{p}$ with the x-axis (see Figure 1 (b)). When $r = 1$, we say the complex number is unit. In this



(a) (b)

Figure 1. Rotation of the plane (a); complex number (b).

way, an unit complex number z can be written in the form

$$z = e^{i\theta_z} = \cos\theta_z + i\sin\theta_z.$$

Hence, the product of $\mathbf{p}$ by the unit complex z is given by

$$\mathbf{p}z = zp = re^{i(\theta_z+\theta_p)}.$$

Therefore, $\mathbf{p}z$ is the point in the plane obtained by rotating $\mathbf{p}$ by an angle θ_z about the origin.

Therefore, we define a biunivoc application $\varphi\colon S^1 \to \mathrm{SO}(2)$, representing a rotation of an angle θ by the complex unit number $z = e^{i\theta}$. More precisely, we associate the the rotation $\varphi(z)$ to the complex number $z \in S^1$ such that $\varphi(z).\mathbf{p} = z\mathbf{p} = \mathbf{p}z$. It is easy to verify that $\varphi(z).\mathbf{p} = z\mathbf{p} = \mathbf{p}z$. In other words, the product of complex numbers in S^1 corresponds to the (composed) product of rotations. Therefore we can work with rotations in the plane using unit complex numbers.

The linear interpolation of two rotations $z_1 = R(\theta_1) = e^{i\theta_1}$ and $z_2 = R(\theta_2) = e^{i\theta_2}$ can be obtained for the complex number, in an elegant way,as follows

$$R(t) = e^{i[(1-t)\theta_1+t\theta_2]}, \qquad t \in [0,1].$$

It is easy to verify that the above interpolation equation can still be written in a briefer algebraic way:

$$R(t) = (z_2 z_1^{-1})^t z_1, \qquad t \in [0,1].$$

The representation of a rotation in the plane by a complex number allows obtaining a parameterization of the space $\mathrm{SO}(2)$ defining the exponential application $\exp\colon \mathbb{R} \to S^1$, for

$$\exp(t) = e^{it} = \cos t + i\sin t.$$

Geometrically, this application is defined in the straight tangent line to S^1 in the point $(0,1)$. It applies each real number t in the extreme point of the arch length $|t|$ going from point $(0,1)$ to point $(\cos t, \sin t)$ in the circle (see Figure 2). Notice that the exponential is biunivoc in the interval $(-\pi, \pi)$

In summary, the representation of rotations by complex numbers is very convenient, and it brings several advantages in relation to matrix representation. Amongst them, we can mention:

- A 2×2 matrix (4 components) occupies more storage space of than a complex number (2 components).
- The number of operations to multiply two complex numbers (4 products) is smaller than the number of operations to multiply a matrix by a vector (8 products).

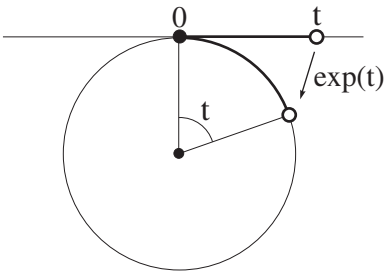


Figure 2. Exponential application.

Finally, we do not obtain a good result using matrix representation to compute interpolation of rotations: the interpolated matrix

$$R(t) = (1-t)\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix} + t\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}.$$

do not represent, in general, even a rotation in the plane.

Those facts show the representation by complex numbers is superior to the representation of rotations by matrices. The study of the space of rotations in $\mathbb{R}^3$ is well more complex than the two-dimensional case seen above. However, we will see that the results of this section admit a good amount of generalization by substituting the complex numbers for another set with a similar algebraic structure: the Hamilton quaternions.

2 Rotations in the Euclidean Space

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis of the Euclidean space $\mathbb{R}^3$ and $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ an orthonormal basis. The change of basis $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{e}_i) = \mathbf{f}_i$ defines an orthogonal transformation. If the determinant of T is positive, then F has the same orientation of E and the transformation T is said to be positive. The space of the positive orthogonal transformations is indicated by $\mathrm{SO}(3)$, and it is called *orthogonal special group* of order 3.

A rigid body does not suffer deformations when moving in the space. Therefore, its positioning is completely determined for a reference frame $\{O, \mathbf{f}_1, \mathbf{f}_1, \mathbf{f}_3\}$ attached to it (local reference frame). The origin O of the reference frame supplies the position of the rigid body, and the basis $\{\mathbf{f}_1, \mathbf{f}_1, \mathbf{f}_3\}$ defines its orientation. Therefore, the orthogonal transformations determine the orientation of a rigid body in space.

Our goal is to understand the space $\mathrm{SO}(3)$. We can state an important first topological result one about this space:

"*The space* $\mathrm{SO}(3)$ *is compact.*"

In fact, it is closed because it is a subset of matrices with determinant 1, and it is limited once the columns of a matrix in $\mathrm{SO}(3)$ have norm equal to 1.

We will look for a better understanding of the topology and of the geometry of that space to obtain good representations and parameterizations. An important result in this direction is given by

Theorem 1. *If R is an element of $\mathrm{SO}(3)$, there is an orthonormal basis of $\mathbb{R}^3$ in which the matrix of R has the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{4.2}$$

Geometrically, the theorem affirms that a positive orthogonal transformation is given by one rotation of an angle θ about an axis (the defined axis for the first vector of the basis). The above theorem is known in Mechanics as Euler's Theorem[1]:

"*Every orientation is defined by a rotation about an axis.*"

Euler's Theorem extends for orthogonal transformations in $\mathbb{R}^n$. A detailed discussion can be found in (Lima, 1999). From now on we will refer to $\mathrm{SO}(3)$ as *space of rotations*.To understand geometrically the rotations of $\mathbb{R}^3$ and to look for tools for a correct manipulation are well more difficult than one could imagine at first view. Let us see one simple example to convince us.

Example 1. The set of rotations $\mathrm{SO}(3)$, with the usual operation of matrix multiplication, has a group structure. In fact, it is easy to verify that the product of positive orthogonal transformations is orthogonal and positive as well. Therefore, if R_1 and R_2 are rotations about an r_1 and r_2 axes, respectively, the product R_1R_2 is also a rotation. Question: *What is the rotation axis of the product?*

3 The space of rotations

Previously, we saw that the space of rotations in the plane is naturally identified with the circle S^1, which is naturally embedded in the $\mathbb{R}^2$. In reality, this is the topological fact that allowed us to obtain a good representation of rotations in the plane. A question appears in a natural way:

"*Can we represent the space $\mathrm{SO}(3)$ by a surface $S \subset \mathbb{R}^3$?*"

We will make a first intuitive attempt for answering the above question. From Euler's theorem, a rotation is determined by an axis, which in turn is completely determined by

[1] Leonhard Euler (1707-1783), Swiss Mathematician.

a unit vector in $\mathbb{R}^3$. Therefore, the set of the axes is naturally identified with the unit sphere in $\mathbb{R}^3$:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}.$$

Besides an axis, we also need a rotation angle, that is one more degree of freedom. We therefore see the space of rotations having dimension 3: two degrees of freedom to choose the axis, and one more degree of freedom to choose the rotation angle about the axis [2].

The fact that space $\mathrm{SO}(3)$ has dimension 3 indicates some difficulty in its study. In fact, in the case of the plaen we saw that $\mathrm{SO}(2)$ has dimension 1 and it is naturally embedded in $\mathbb{R}^2$. It is topological fact that the tridimensionality of $\mathrm{SO}(3)$, together with his fact it is compact, shows that it cannot be embedded in $\mathbb{R}^3$. This means that, topologically, the rotations of the $\mathbb{R}^3$ space do not belong to $\mathbb{R}^3$ and we should look for one representation in spaces of larger dimension.

We begin our search with the matrix representation of $\mathrm{SO}(3)$.

3.1 Representation by matrices with linkage

We know that a rotation that takes the canonical basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$, in another positive orthonormal basis $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ becomes completely determined by the matrix

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix},$$

where $T(\mathbf{e}_i) = \sum_{j=1}^{3} f_{ji}\mathbf{f}_j$.

Therefore, in a first inspection, we can parameterize the space of orientations using 3×3 matrices. In this case, ee would have a representation naturally embedded in the $\mathbb{R}^9$ (9 degrees of freedom). However, it occurs that the columns of the above matrix are the coordinates of the vectors $\mathbf{f}_i$ that form an orthonormal basis. Therefore,

$$|\mathbf{f}_j|^2 = \sum_{j=1}^{3} f_{ij}^2 = 1, \quad j = 1, 2, 3. \tag{4.3}$$

and

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle = \sum_{i=1}^{3} f_{ij} f_i k = 0, \quad j \neq k, j, k = 1, 2, 3. \tag{4.4}$$

These six equations reduce the degrees of freedom of choosing the matrix of T from 9 for 3. (In reality, we are defining the space of rotations implicitly in $\mathbb{R}^9$.)[3] Working

[2] In an intuitive way, we have two degrees of freedom by pointing our arm in a direction in space, we still have a degree of freedom by rotating the arm longitudinally.

[3] Note we have here another demonstration that $\mathrm{SO}(3)$ has dimension 3.

with such matrix representation has always the problem of guaranteeing that the matrices under consideration satisfy to the six linkage conditions from equations (4.3) and (4.4). That is a difficult problem and computationally costly.

We will continue with our search by describing a local parameterization of the space of rotations which is very used in the Mechanics, and it was discovered by Leonhard Euler.

4 Parameterization by Euler's angles

Consider the canonical basis $\mathcal{E} = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, as shown in Figure 3(a). This reference frame defines the Cartesian coordinates (x, y, z) in space. We will apply three consecutive rotations in this reference frame about the coordinate axes z, x and z of the canonical basis, as described below:

$R_z(\psi)$: Rotation of an angle ψ about the z-axis (see Figure 3(b));

$R_x(\theta)$: Rotation of an angle θ about the x-axis (see Figure 3(c));

$R_z(\phi)$: Another rotation about the z-axis by an angle ϕ (see Figure 3(d)).

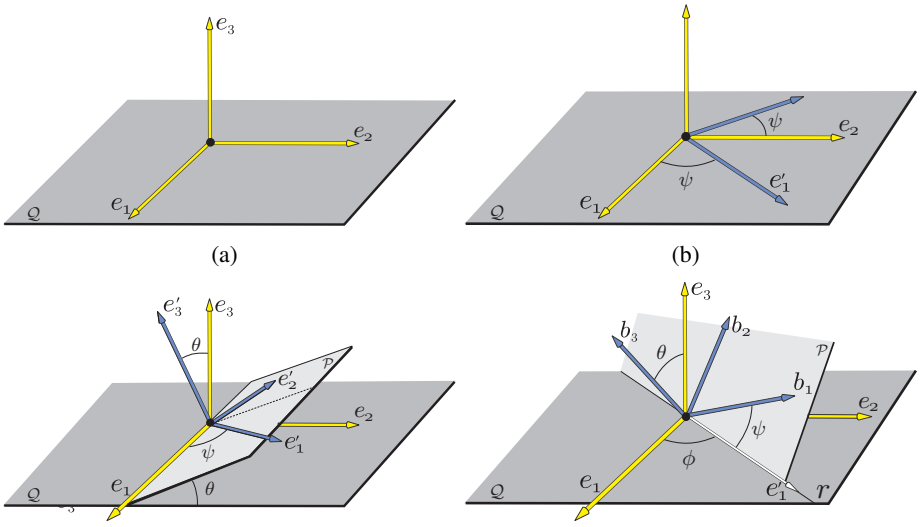


Figure 3. Euler's Angles.

The product

$$R_z(\psi)R_x(\theta)R_z(\phi), \tag{4.5}$$

of the above rotations transforms the canonical basis $\mathcal{E}$ into the orthonormal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ shown in Figure 3(d).

Now let us suppose the orthonormal reference frame $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is given a priori. Statement: " *the reference frame $\mathcal{B}$ can be obtained from the canonical basis $\mathcal{E}$ by three consecutive rotations* ". In fact, as we showed in Figure 3(d), let $\mathcal{P}$ be the plane generated by the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of the basis $\mathcal{B}$, and r the intersection straight line between $\mathcal{P}$ with the plane $z = 0$ (r is called *nodal straight line* in Mechanics). Notice that the unit vector $\mathbf{e}_3 \wedge \mathbf{b}_3$ points towards the direction of the nodal straight line. (In fact, $\mathbf{e}_3$ and $\mathbf{b}_3$ are normal to the planes $z = 0$ and $\mathcal{P}$, respectively. Therefore, the cross product $\mathbf{e}_3 \wedge \mathbf{b}_3$ should point towards the direction of the intersection straight line between these two planes, which is the nodal straight line.) Let us also indicate by ϕ the angle between the vector $\mathbf{e}_1$ and the nodal straight line.

Initially we do a rotation $R_{e_3}(\phi)$, of an angle ϕ about the axis e_3, that takes the vector $\mathbf{e}_1$ into the vector $\mathbf{e}_1' = \mathbf{e}_3 \wedge \mathbf{b}_3$. Soon afterwards, we apply the rotation $R_{\mathbf{e}_3 \wedge \mathbf{b}_3}(\theta)$, by an angle θ about the nodal straight line, which takes the vector e_3 into the vector $\mathbf{b}_3$. Finally, we apply the rotation $R_{\mathbf{b}_3}(\psi)$, by an angle ψ about the vector $\mathbf{b}_3$ which takes the vector $\mathbf{e}_1'$ into the vector $\mathbf{b}_1$. Of course, this sequence of rotations

$$R_{\mathbf{e}_3}(\phi) R_{\mathbf{e}_3 \wedge \mathbf{b}_3}(\theta) R_{\mathbf{b}_3}(\psi), \tag{4.6}$$

transforms the canonical basis into the basis $\mathcal{B}$.

Notice that the transformation in (4.5) is equal to the one in (4.6) because both coincide in the canonical basis $\mathcal{E}$. However, they apply the rotations in inverse order and with different axes. In the first case, the rotations are always done around the axes of the Cartesian (global) system; In the second case, the rotations are applied consecutively in local coordinate systems (except for $R_{\mathbf{e}_3}(\phi)$, of course).

We know that a rotation R in space is determined for the positive orthonormal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. (The rotation is given by the linear transformation taking the canonical basis $\mathcal{E} = \{\rceil_1, \rceil_2, \rceil_3\}$ into the basis $\mathcal{B}$). Therefore, we showed above the following result:

> "*A rotation $R \in \mathrm{SO}(3)$ can be obtained by three consecutive rotations about the coordinate axes.*"

This result is owed to Leonhard Euler, and the angles (ϕ, θ, ψ) are called *Euler's angles*. They constitute a parameterization of the space $\mathrm{SO}(3)$. This parameterization is used broadly in the study of rigid bodies dynamics in $\mathbb{R}^3$.

4.1 Yaw, Pitch and Roll

In the previous section, we saw two different choices of parameterizations by Euler's angles, with different rotation axes and order about them. The parameterization of $\mathrm{SO}(3)$ using Euler's angles is broadly used in the Computer Graphics aiming at specifying rotations. It is common to use names brought from aviation: yaw (deflection angle),

pitch (steepness angle), roll (longitudinal rotation angle), as shown in Figure 4(a). In this convention, the roll, yaw and pitch axes are the z, x, and y axes, respectively (Figure 4(b)).

4.2 Euler's angles and matrices

We are going to describe a rotation matrix parameterized by Euler's angles. Indicating the rotation angles about the $]_1$, $]_2$ and $]_3$ axes for α, β and γ, respectively, gives us the rotation matrices at each axis:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

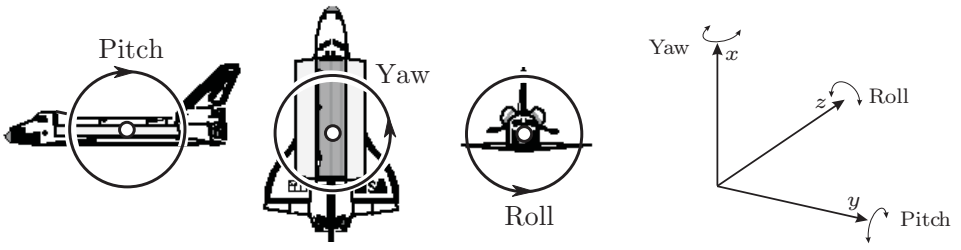
$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Making the product of these three matrices, in the order $R_z(\gamma)R_y(\beta)R_x(\alpha)$ (which is the order used to obtain equation (4.6)), we obtain the parameterization $R(\alpha, \beta, \gamma)$ of the rotation matrix,

$$\begin{pmatrix} \cos\beta\cos\gamma & \cos\gamma\sin\alpha\sin\beta - \cos\alpha\sin\gamma & \cos\alpha\cos\gamma\sin\beta + \sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & -\cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma \\ -\sin\beta & \cos\beta\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}. \quad (4.7)$$

which is called *Euler's matrix*. Notice that R defines an application $R\colon \mathbb{R}^3 \to \mathbb{R}^9$.



(a) (b)

Figure 4. Euler's Angles.

4.3 Singularities and Euler's angles

Euler's angles do not constitute a global parameterization of the space of rotations. The reason being the space $\mathrm{SO}(3)$ is compact, and therefore it cannot be parameterized by an open set of $\mathbb{R}^3$ (in the same way a sphere $S^2 \subset \mathbb{R}^3$ does not admit a global parameterization for an open subset in the plane). In this way, any attempt of extending Euler's angles to cover the whole space of rotations leads to the creation of singularities in the parameterization. These singularities are regions in the domain of the parameterization in which we do not have the three degrees of freedom in the rotation matrix (see the discussion of singularity in the parameterization of the sphere in the chapter on coordinate systems).

We can describe singularities of the parameterization for Euler's angles by taking $\beta = \pi/2$ in the parameterization $R(\alpha, \beta, \gamma)$ of equation (4.7). That is, doing a rotation of 90° about the y-axis (pitch), we obtain the parameterization

$$\begin{aligned} R(\alpha, \frac{\pi}{2}, \gamma) &= \begin{pmatrix} 0 & \cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & 0 \\ 0 & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\gamma - \cos\gamma\sin\alpha & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \\ 0 & \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We see that, despite having two degrees of freedom in the parameter space, we have just one degree of freedom in the parametrization. This is because the parametrized matrix only depends on the difference of the angles. (This phenomenon is similar to the singularity in the parameterization of the sphere that we saw in the chapter of coordinate system.)

It is easy to understand intuitively the singularities of the parameterization of $\mathrm{SO}(3)$ using Euler's angles. In fact, the angles perform three consecutive and independent rotations in each one of the coordinate axes: first we rotate about z, then about y, and later about x. The singularities happen as two axes of rotation point towards the same direction. This is because the rotations about those two axes are dependent and therefore we loose a degree of freedom.

The gimbal lock phenomenon

The problem of singularities of Euler's angles manifests in practice. The phenomenon is known in Aeronautical Engineering and Mechanical Engineering as *gimbal lock* [4].

[4] Gimbal is the given name to the mechanical assembly of equipments requiring a space orientation and, for that end, using Euler's angles: for example, gyroscope, optic equipments

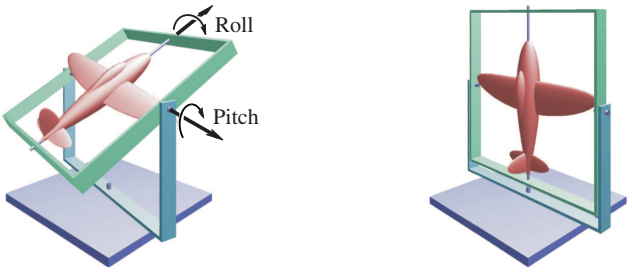


Figure 5. Gimbal lock phenomenon.

Notice the example in the model shown in Figure 5. As the yaw-axis is fixed, after a pitch-axis rotation (leaving the airplane heading in the vertical direction), the roll and yaw axes coincide. In this position we loose a degree of freedom in the rotation about those two axes.

Of course, by restricting the variation of the Euler's angles, we eliminate the singularities problem. For this reason, the pilot of an airplane does not have the same problems of an animator (fortunately!).

5 Interpolation of rotations

We already had the opportunity to discuss in several occasions the importance of interpolation methods in Computer Graphics. They are responsible for the reconstruction of the objects starting from a set of samples.

The general problem of interpolating rotations consists in interpolating points in the space $\mathrm{SO}(3)$, and it can be stated in the following way:

> "*Given n rotations $R_0, R_1, \ldots, R_n$, obtain a curve $R\colon [0,1] \to \mathrm{SO}(3)$, of class C^k, $k \geq 0$, and a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ in the interval $[0,1]$, such that $R(0) = R_0$, $R(1) = R_1$ and $R(t_i)$ are an approximation of the rotation R_i for $i = 1, \ldots, n-1$.*"

In reality, in pure interpolation methods, we require that $R(t_i) = R_i$, for $i = 0, 1, \ldots, n$. The interpolation of rotations play a fundamental part in the *keyframe* aniamtion technique used in the majority of the commercial animation systems. In those systems, the animator specifies the position of the rigid objects (translation+rotation) at some points, the keyframes, and the system computes the interpolation to rebuild the motion.

An important particular case of the interpolation problem, consists on obtaining the interpolation between two rotations R_0 and R_1. As the space holds an affine structure (e.g. the Euclidean space $\mathbb{R}^n$), the natural and simple method for obtaining an interpolation

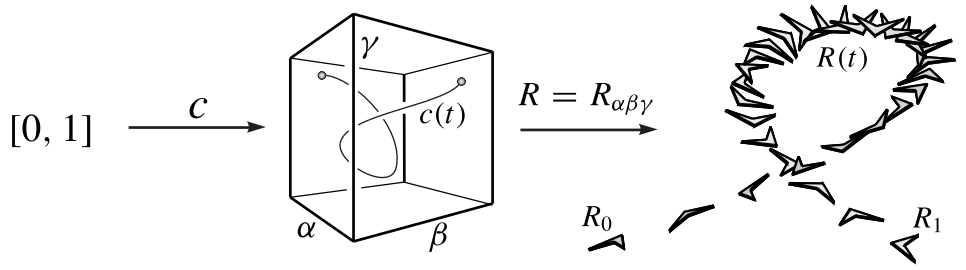


Figure 6. Moving a boomerang by interpolating Euler's angles.

between two objects R_0 and R_1 is a *linear interpolation*, defined as $R(t) = (1-t)R_0 + tR_1, t \in [0,1]$.

In the case R_0 and R_1 are matrices, then linear interpolation makes sense, because the space of matrices is a vector space. However, if R_0 and R_1 are rotation matrices, the matrix $R(t) = (1-t)R_0 + tR_1, t \in [0,1]$, in general does not represent a rotation. This happens because the space of rotations $\mathrm{SO}(3)$ is not a linear subspace of matrix space. (The problem here is similar to a linear interpolation between two points on a sphere: the resulting segment is not contained in the sphere.)

5.1 Interpolating Euler's angles

When we have a parameterization $\varphi\colon U \subset \mathbb{R}^m \to S$ of a space S, the interpolation problem is reduced to interpolate in the parameters space U. In fact, to interpolate n points $\mathbf{p}_1, \ldots, \mathbf{p}_n \in S$, we take the corresponding points $\varphi^{-1}(\mathbf{p}_1), \ldots, \varphi^{-1}(\mathbf{p}_n)$, in the parameters space $U \subset \mathbb{R}^n$, and we obtain an interpolation curve $c\colon [0,1]\colon \to \mathbb{R}^m$ among those points. The desired interpolation curve is given by the composite $\varphi \circ c\colon [0,1] \to S$. Therefore, the interpolation problem is reduced to one in $\mathbb{R}^m$, which is broadly covered in the literature.

In particular, the above principle is applied to interpolate rotations using the parameterization by Euler's angles as shown in Figure 6. In this case, if we have two rotations $R_0 = R(\alpha_0, \beta_0, \gamma_0)$ and $R_1 = R(\alpha_1, \beta_1, \gamma_1)$, we obtain a curve c in the parameters space, $c(t) = (\alpha(t), \beta(t), \gamma(t))$, such that $c(0) = (\alpha(0), \beta(0), \gamma(0)) = R_0$ and $c(1) = (\alpha(1), \beta(1), \gamma(1)) = R_1$. The interpolating path in $\mathrm{SO}(3)$ is given by the composite $R(t) = R(\alpha(t)), R(\beta(t)), R(\gamma(t))$, where $R = R_{\alpha\beta\gamma}$ is the Euler matrix given in the equation (4.7). (we can of course use this method to interpolate n rotations.)

The advantage of this method is that the interpolation itslef is done in the $\mathbb{R}^3$ space. In this way, we can interpolate an arbitrary number of rotations using the various point interpolation methods of interpolation described in the literature (e.g. Splines, Bezier, Hermite). This method is, of course, applicable to any other parameterization of the space

of rotations.

The interpolation of Euler's angles is the method used in the majority of keyframe animation systems to specify the orientation of the objects and of the virtual camera. The disadvantages of this method are all those inherent to the parameterization method with Euler's angles that we will discuss next.

6 Pause for commercials

The parameterization of the space of rotations using Euler's angles is broadly used in the area of Animation. A sequence of rotations with Euler's angles is represented by a three-dimensional curve where each component represents a rotation angle in each axis. Therefore, the animators work by manipulating curves in the $\mathbb{R}^3$ space to modify path, speed, acceleration etc. In general, the editing is made over the projection of these curves in the coordinate planes, an interface in which animators are already quite familiar with.

Another advantage of matrices, and in particular of Euler matrices, is that other projective transformations are also represented by matrices, especially translation.

However, besides the problem of singularities (gimbal lock), discussed above, the parameterization for Euler's angles has several other inconvenients.

Given a rotation matrix, we can have ambiguities in the solution of the inverse problem. What are the axis and rotation angle?

The representation for Euler's angles has no unicity because it depends on the order of the axes (anisotropy);

The parameterization defines a local coordinate system in the space of rotations; In other words, we cannot describe all rotations using just an Euler coordinate system. Of course, we can use several parameterizations to cover the $\mathrm{SO}(3)$ space; However, it is not simple to perform a change of coordinates between two systems of Euler's angles.

Finally, specifying rotations using Euler's angles is difficult and not intuitive. Let us see a practical example: hold a football ball, mark a point on it with a pen and rotate it to place the marked point in the vertical position (north pole of the ball). Easy, right? Now try to solve the same problem with three successive rotations about the axes according to Euler's parameterization. In the first solution, in an intuitive and natural way, we used Euler's Theorem, by doing a single rotation about an axis.

The above fact shows that we should look for a representation of $\mathrm{SO}(3)$ where each rotation is represented by its axis, together with the rotation angle. To have this intuitive representation mathematically efficient, we need to introduce a mathematical structure in space constituted by the elements " axis + rotation angle ". This will be our goal in the remaining of this chapter.

7 Quaternions

Quaternions introduce an algebraic structure in the Euclidean space $\mathbb{R}^4$ similar to the structure of the complex numbers in the $\mathbb{R}^2$ plane. This structure allows to obtain a representation of the space of rotations based on the " axis+angle "model to define a rotation.

We denote as $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the canonical basis of the space $\mathbb{R}^4$: $\mathbf{1} = (1, 0, 0, 0)$, $\mathbf{i} = (0, 1, 0, 0)$, $\mathbf{j} = (0, 0, 1, 0)$ and $\mathbf{k} = (0, 0, 0, 1)$. A point $\mathbf{p} = (w, x, y, z) \in \mathbb{R}^4$ is then written in the form $\mathbf{p} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Amongst the four elements $\mathbf{1}, \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ of the basis, we define a *product* according to the table below:

$\cdot$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	**-1**	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	**-1**	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	**-1**

Table 1: Multiplication of the quaternions in the basis.

Notice that the product is not commutative. We extended the above product to obtain the product $\mathbf{pq}$ between two arbitrary points of $\mathbb{R}^4$ requiring bilinearity. In other words, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^4$, we have

$$(a\mathbf{p}_1 + \mathbf{p}_2)(b\mathbf{q}_1 + \mathbf{q}_2) = ab\mathbf{p}_1\mathbf{q}_1 + a\mathbf{p}_1\mathbf{q}_2 + b\mathbf{p}_2\mathbf{q}_1 + \mathbf{p}_2\mathbf{q}_2.$$

The space $\mathbb{R}^4$, with the above multiplicative structure, is called *quaternions space* or simply *quaternions*. Quaternions were discovered by William Hamilton[5] with the goal of obtaining in the space the same relation between rotations and complex numbers that exists in the plane. The operation of quaternion product is distributive, associative (tedious check), and, highlighting once again, not commutative.

The subspace $\mathbb{R}\,\mathbf{1} = \{(x, 0, 0, 0); x \in \mathbb{R}\}$ is identified with the set of the real numbers, and it is called *space of scalars*; the subspace $\mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k} = \{(0, x, y, z); x, y, z \in \mathbb{R}\}$ is naturally identified with the Euclidean space $\mathbb{R}^3$ and it is called *space of vectors* or *space of pure quaternions*. Given a quaternion $\mathbf{u} \in \mathbb{R}^4$, $\mathbf{u} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the number w is called *real part* of $\mathbf{u}$ and it is indicated by $\Re(\mathbf{u})$; the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, is called of *vector part* of $\mathbf{u}$. If $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ is a vector, we indicate by $\hat{v}$ the associated pure quaternion: $\hat{\mathbf{v}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. In this way, every quaternion can be written in the form $\mathbf{p} = w + \hat{\mathbf{v}}$, $w \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$.

[5] Sir William Rowan Hamilton (1805-1865), Irish mathematician.

The *conjugated* of a quaternion $\mathbf{u} = w + \hat{\mathbf{v}}$ is defined by $\mathbf{u}^\star = w - \hat{\mathbf{v}}$. In other words, if $\mathbf{u} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{u}^\star = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. The conjugation operation satisfies the properties $(\mathbf{u}^\star)^\star = \mathbf{u}$ and $(\mathbf{uv})^\star = \mathbf{u}^\star\mathbf{v}^\star$. Of course, the real part of a quaternion $\mathbf{u}$ is equal to $\Re(\mathbf{u}) = (\mathbf{u} + \mathbf{u}^\star)/2$.

The *norm* of a quaternion $\mathbf{p} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is equal to the norm of the vector (w, x, y, z) in the Euclidiano inner product of $\mathbb{R}^4$:

$$|\mathbf{p}| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Of course if $\mathbf{p} = w + \hat{\mathbf{v}}$ then $|\mathbf{p}|^2 = w^2 + |\hat{\mathbf{v}}|^2$.

Also, $|\mathbf{p}| = |\mathbf{p}^\star|$. The norm behaves well in relation to the quaternion product (verify):

$$|\mathbf{p}_1\, \mathbf{p}_2| = |\mathbf{p}_1||\mathbf{p}_2|. \tag{4.8}$$

If $|\mathbf{p}| = 1$, we say $\mathbf{p}$ is a *unit quaternion*. Of course, the set of unit quaternions form the unit sphere $S^3 \subset \mathbb{R}^4$,

$$S^3 = \{(w, x, y, z) \in \mathbb{R}^4; w^2 + x^2 + y^2 + z^2 = 1\}.$$

Besides, for (4.8), the product of two unit quaternions is a unit quaternion. Therefore, S^3 is a group, as seem as the set of unit quaternions. (Notice here the analogy with the circle $S^1 \subset \mathbb{R}^2$ that is a group formed by the complex numbers of norm 1.)

The *multiplicative inverse* of a quaternion $\mathbf{u}$, is a quaternion $\mathbf{u}^{-1}$ such that $\mathbf{u}\mathbf{u}^{-1} = \mathbf{u}^{-1}\mathbf{u} = \mathbf{1}$. An immediate calculation shows that every nonull quaternion $\mathbf{u}$ has one multiplicative inverse given by

$$\mathbf{u}^{-1} = \frac{\mathbf{u}^\star}{|\mathbf{u}|^2},$$

Notice if $|\mathbf{u}| = 1$, then $\mathbf{u}^{-1} = \mathbf{u}^\star$.

If $\mathbf{q}_1 = a_1 + \hat{\mathbf{v}}_1$ and $\mathbf{q}_2 = a_2 + \hat{\mathbf{v}}_2$, a direct calculation shows that the product of $\mathbf{q}_1$ and $\mathbf{q}_2$ is given by

$$\mathbf{q}_1\mathbf{q}_2 = a_1a_2 - \langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \rangle + a_1\hat{\mathbf{v}}_2 + a_2\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2.$$

We are using the notation $\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2$ to indicate $\widehat{\mathbf{v}_1 \wedge \mathbf{v}_2}$. Therefore, if $\mathbf{q}_1$ and $\mathbf{q}_2$ are pure quaternions, that is, $a_1 = a_2 = 0$, then

$$\mathbf{q}_1\mathbf{q}_2 = -\langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \rangle + \hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2.$$

Therefore, the quaternion product synthesizes two types of vector products in $\mathbb{R}^3$: the scalar and cross products (in a way this let us foresee the great potential of this operation). Notice if $\mathbf{v}$ is a pure quaternion, $\mathbf{v} \wedge \mathbf{v} = 0$, and by taking $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{v}$ in the previous equation, we obtain

$$\mathbf{v}^2 = -\langle \mathbf{v}, \mathbf{v} \rangle = -||\mathbf{v}||^2.$$

In particular, if $\mathbf{v}$ is a unit pure quaternion, we have $\mathbf{v}^2 = -1$.

Every quaternion $q = a + \hat{\mathbf{v}}$ can be written in the form $a + b\hat{\mathbf{u}}$, where $|\mathbf{u}| = 1$ is a unit quaternion. In fact, we can only take

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad \text{e} \quad b = |\mathbf{v}|.$$

Observe that the representation of a quaternion in the form $\mathbf{q} = a + b\hat{\mathbf{u}}$, as $|\hat{\mathbf{u}}| = 1$, that is, $\mathbf{u}$ is a pure unit quaternion, we have $\mathbf{u}^2 = -1$. Yet, in this representation, if $\mathbf{q}$ is unit, we have

$$1 = |q|^2 = a^2 + b^2|\hat{\mathbf{u}}|^2 = a^2 + b^2.$$

Therefore, there exists $\theta \in \mathbb{R}$ such that $a = \cos\theta$ and $b = \sin\theta$, and we can write $\mathbf{q} = \cos\theta + \sin\theta\,\hat{\mathbf{u}}$.

Notice the similarity of what we have obtained above with the complex numbers. A complex number has the form $x + iy$ with $i^2 = -1$; Besides, every unit complex number, i.e. of norm 1, can be written in the form $\cos\theta + i\sin\theta$. The difference is that, unlike the complex number i, the unit quaternion $\hat{\mathbf{u}}$, in the representation $p = a + b\hat{\mathbf{u}}$, depends on the quaternion $\mathbf{p}$.

The results here show that quaternions have algebraic properties similar to the ones in complex numbers, with exception for commutativity. Our final test is to try to extend the result involving complex numbers and rotations of the plane to quaternions. This is the subject of the next section.

7.1 Representation of rotations by quaternions

Given a unit quaternion $\mathbf{q}$, we should define the transformation $R_{\mathbf{q}}\colon \mathbb{R}^4 \to \mathbb{R}^4$ placing $R_{\mathbf{q}}(\mathbf{p}) = \mathbf{pq}$. We have $|R_{\mathbf{q}}(\mathbf{p})| = |\mathbf{p}||\mathbf{q}| = |\mathbf{p}|$, therefore the transformation $\mathbb{R}_{\mathbf{q}}$ is orthogonal. That is, multiplications for unit quaternions geometrically correspond to rotations in $\mathbb{R}^4$ space. Now, a natural question comes into play:

"*How to obtain a representation of* $\mathrm{SO}(3)$ *by quaternions?*"

A first idea would be to associate, to each quaternion $\mathbf{q} \in \mathbb{R}^3$, the transformation $R_{\hat{\mathbf{q}}}$. Unfortunately, in general, $R_{\hat{\mathbf{q}}} \notin \mathrm{SO}(3)$, that is, the space of pure quaternions, naturally identified with $\mathbb{R}^3$, is not invariant for $R_{\hat{\mathbf{q}}}$. The representation of $\mathrm{SO}(3)$ by quaternions is a little subtler as we will see.

Observe that the representation of a unit quaternion $\mathbf{q}$ in the form $\mathbf{q} = \cos\theta + \sin\theta\,\hat{\mathbf{u}}$, with $|\hat{\mathbf{u}}| = 1$, shows this quaternion defines an angle θ and an axis $\hat{\mathbf{u}}$ of the space. It is natural to expect this quaternion to represent a rotation about the $\mathbf{u}$-axis . We will show this in fact happens and the angle of that rotation is 2θ.

Given a unit quaternion $\mathbf{q}$, we define the transformation $\varphi_{\mathbf{q}}\colon \mathbb{R}^3 \to \mathbb{R}^3$, placing

$$\varphi_{\mathbf{q}}(\mathbf{v}) = \mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{-1} = \mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{\star}, \qquad \mathbf{v} \in \mathbb{R}^3. \tag{4.9}$$

We will demonstrate several properties soon afterwards of that transformation.

the transformation $\varphi_{\mathbf{q}}$ is well defined. This means $\varphi_{\mathbf{q}}(\mathbf{v}) \in \mathbb{R}^3$. For that, it is enough to show $\Re(\varphi_{\mathbf{q}}(\mathbf{v})) = 0$:

$$\begin{aligned}\Re(\mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{-1}) &= \Re(\mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{\star});\\ &= [\mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{\star} + (\mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{\star})^{\star}]/2;\\ &= [\mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{\star} + \mathbf{q}\hat{\mathbf{v}}^{\star}\mathbf{q}^{\star}]/2;\\ &= \mathbf{q}[(\hat{\mathbf{v}} + \hat{\mathbf{v}}^{\star})/2]\mathbf{q}^{\star};\\ &= \mathbf{q}\Re(\hat{\mathbf{v}})\mathbf{q}^{\star} = \Re(\hat{\mathbf{v}})\mathbf{q}\mathbf{q}^{\star} = \Re(\hat{\mathbf{v}}) = 0.\end{aligned}$$

The transformation $\varphi_{\mathbf{q}}$ is linear. In fact,

$$\begin{aligned}\varphi_{\mathbf{q}}(a\hat{\mathbf{u}} + \hat{\mathbf{v}}) &= q(a\hat{\mathbf{u}} + \hat{\mathbf{v}})q^{\star}\\ &= qa\hat{\mathbf{u}}q^{\star} + q\hat{\mathbf{v}}q^{\star}\\ &= a(q\hat{\mathbf{u}}q^{\star}) + (q\hat{\mathbf{v}}q^{\star})\\ &= a\varphi_{\mathbf{q}}(\mathbf{u}) + \varphi_{q}(\mathbf{v}).\end{aligned}$$

The transformation $\varphi_{\mathbf{q}}$ is orthogonal. In fact,

$$|\varphi_{\mathbf{q}}(\mathbf{v})| = |q\hat{\mathbf{v}}q^{\star}| = |q||\hat{\mathbf{v}}||q^{\star}| = |\mathbf{v}|.$$

The transformation $\varphi_{\mathbf{q}}$ is a rotation of $\mathbb{R}^3$. In fact, as we saw above, it is enough to show $\varphi_{\mathbf{q}}$ is positive. In this case, notice for $\mathbf{q} = 1$ $\varphi_{\mathbf{q}} = I$, where $I\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the identity transformation. As the space of unit quaternions is connected (because it is the unit sphere in $\mathbb{R}^4$), we conclude by a continuity argument that $\varphi_{\mathbf{q}}$ is positive for every quaternion $\mathbf{q}$. *The axis of rotation of $\varphi_{\mathbf{q}}$ is the quaternion $\hat{\mathbf{u}}$.* In fact, it is enough show that $\varphi_{\mathbf{q}}(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$:

$$\begin{aligned}\varphi_{\mathbf{q}}(\mathbf{u}) &= q\hat{\mathbf{u}}q^{\star}\\ &= (\cos\theta + \hat{\mathbf{u}}\sin\theta)\,\hat{\mathbf{u}}\,(\cos\theta - \hat{\mathbf{u}}\sin\theta)\\ &= (\cos\theta)^2\hat{\mathbf{u}} - (\sin\theta)^2\hat{\mathbf{u}}^3\\ &= (\cos\theta)^2\hat{\mathbf{u}} - (\sin\theta)^2(-\hat{\mathbf{u}})\\ &= \mathbf{u}.\end{aligned}$$

The angle of rotation of $\varphi_{\mathbf{q}}$ is 2θ. We will make a geometric demonstration of this fact, by solving the following problem:

> "*Given a rotation R of an angle θ about an axis r, how can we obtain the quaternion $\mathbf{q}$ of the representation $\varphi_{\mathbf{q}}$ in R?*"

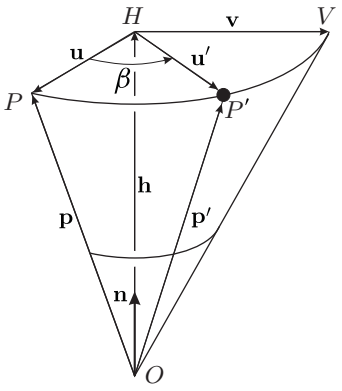


Figure 7. Rotation of an angle θ about the axis defined by vector $\mathbf{n}$

To answer the above question, consider Figure 7 showing the rotation R of point P by an angle β about the OH-axis. The image of point P is the point $P' = R(P)$.

Let us take the positive reference frame $\{\mathbf{u}, \mathbf{v}, \mathbf{n}\}$, where $\mathbf{u} = \overrightarrow{HP}$, $\mathbf{v} = \overrightarrow{HV}$ and $\mathbf{n}$ is the unit vector in the direction of the rotation axis. We also define the vectors

$$\mathbf{h} = \overrightarrow{OH}, \quad \mathbf{p} = \overrightarrow{OP}, \quad \mathbf{p}' = \overrightarrow{OP'}, \quad \text{and} \quad \mathbf{u}' = \overrightarrow{HP'}.$$

with $\mathbf{u} \perp \mathbf{h}$, $\mathbf{h} \perp \mathbf{v}$ and $|\mathbf{v}| = |\mathbf{u}|$. Having $\mathbf{h}$ as the orthogonal projection of $\mathbf{p}$ in the direction $\mathbf{n}$, we have $\mathbf{h} = \langle \mathbf{p}, \mathbf{n} \rangle \mathbf{n}$. It is also easy to verify that

$$\begin{aligned} \mathbf{p}' &= \mathbf{h} + \mathbf{u}' \\ \mathbf{u} &= \mathbf{p} - \mathbf{h} = \mathbf{p} - \langle \mathbf{p}, \mathbf{n} \rangle \mathbf{n} \\ \mathbf{u}' &= \cos\beta \mathbf{u} + \sin\beta \, \mathbf{v}, \end{aligned}$$

and besides,

$$\mathbf{v} = \mathbf{n} \wedge \mathbf{u} = \mathbf{n} \wedge (\mathbf{p} - \mathbf{h}) = \mathbf{n} \wedge (\mathbf{p} - \langle \mathbf{p}, \mathbf{n} \rangle) = \mathbf{n} \wedge \mathbf{p}.$$

We then have

$$\begin{aligned} R(P) = \overrightarrow{OP'} &= \mathbf{h} + \mathbf{u}' \\ &= \langle \mathbf{n}, \mathbf{p} \rangle \mathbf{n} + \cos\beta \mathbf{u} + \sin\beta \mathbf{v} \\ &= \langle \mathbf{n}, \mathbf{p} \rangle \mathbf{n} + \cos\beta (\mathbf{p} - \langle \mathbf{p}, \mathbf{n} \rangle \mathbf{n}) + \sin\beta \, \mathbf{n} \wedge \mathbf{p} \\ &= \cos\beta \mathbf{p} + (1 - \cos\beta) \langle \mathbf{n}, \mathbf{p} \rangle \mathbf{n} + \sin\beta \, \mathbf{n} \wedge \mathbf{p}. \end{aligned} \tag{4.10}$$

We will now calculate $R(P)$ using the rotation $\varphi_{\mathbf{q}}$ in (4.9). In this case point P is represented by the pure quaternion, $P = \hat{\mathbf{p}}$, and $\mathbf{q}$ is the unit quaternion $\mathbf{q} = \cos\theta +$

$\hat{\mathbf{n}}\sin\theta$. To simplify the calculations let us take $c = \cos\theta$ and $\mathbf{t} = \hat{\mathbf{n}}\sin\theta$, therefore $\mathbf{q} = c + \hat{\mathbf{t}}$. We then have

$$\begin{aligned} R(P) = \varphi_{\mathbf{q}}(\mathbf{p}) &= \mathbf{q}\hat{\mathbf{p}}\mathbf{q}^{\star} \\ &= (c+\hat{\mathbf{t}})(\hat{\mathbf{p}})(c-\hat{\mathbf{t}}) \\ &= (c+\hat{\mathbf{t}})(\langle\hat{\mathbf{p}},\hat{\mathbf{t}}\rangle - \hat{\mathbf{p}}\wedge\hat{\mathbf{t}} + c\hat{\mathbf{p}}) \\ &= c\langle\hat{\mathbf{p}},\hat{\mathbf{t}}\rangle - \langle\hat{\mathbf{t}}, -\hat{\mathbf{p}}\wedge\hat{\mathbf{t}}\rangle - c\langle\hat{\mathbf{t}},\hat{\mathbf{p}}\rangle + \\ &\quad + \hat{\mathbf{t}}\wedge(-\hat{\mathbf{p}}\wedge\hat{\mathbf{t}}) + c(\hat{\mathbf{t}}\wedge\hat{\mathbf{p}}) \quad + c(-\hat{\mathbf{p}}\wedge hat\mathbf{t} + c\hat{\mathbf{p}}) + \langle\hat{\mathbf{p}}, hat\mathbf{t}\rangle hat\mathbf{t}) \\ &= \langle hat\mathbf{t},\hat{\mathbf{p}}\rangle hat\mathbf{t} - \langle hat\mathbf{t}, hat\mathbf{t}\rangle\hat{\mathbf{p}} + 2c(hat\mathbf{t}\wedge\hat{\mathbf{p}}) + c^2\hat{\mathbf{p}} + \langle\hat{\mathbf{p}}, hat\mathbf{t}\rangle hat\mathbf{t}) \\ &= (c^2 - \langle hat\mathbf{t}, hat\mathbf{t}\rangle)\hat{\mathbf{p}} + 2\langle\hat{\mathbf{p}}, hat\mathbf{t}\rangle hat\mathbf{t} + 2c(hat\mathbf{t}\wedge\hat{\mathbf{p}}). \end{aligned} \tag{4.11}$$

Substituting the values of $c = \cos\theta$ and $\mathbf{t} = \hat{\mathbf{n}}\sin\theta$, in the expression of $\varphi_{\mathbf{q}}(\mathbf{p})$ obtained in equation (**??**), we have

$$\begin{aligned} R(P) &= (\cos^2\theta - \sin^2\theta)\hat{\mathbf{p}} + 2\sin^2\theta\langle\hat{\mathbf{p}},\hat{\mathbf{n}}\rangle\hat{\mathbf{n}} + 2\sin\theta\cos\theta(\hat{\mathbf{n}}\wedge\hat{\mathbf{p}}) \\ &= \cos 2\theta\hat{\mathbf{p}} + (1-\cos 2\theta)\langle\hat{\mathbf{n}},\hat{\mathbf{p}}\rangle\mathbf{n} + \sin 2\theta(\mathbf{n}\wedge\hat{\mathbf{p}})). \end{aligned}$$

Comparing this equation with (4.10) we see that $\varphi_{\mathbf{q}}(\mathbf{p})$ is a rotation of an angle 2θ about the axis defined by the unit vector $\mathbf{n}$.

The results demonstrated above can be summarized in

Theorem 2 (Representation of rotations for quaternions)**.** *A rotation of an angle 2θ about an axis defined by a unit vector $\mathbf{n}$ is represented by $\varphi_{\mathbf{q}}(\mathbf{v}) = \mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{-1}$, where $\mathbf{q} = \cos\theta + \hat{\mathbf{n}}\sin\theta$.*

Therefore, we have a transformation $\varphi\colon S^3 \to \mathrm{SO}(3)$, $\varphi(\mathbf{q}) = \varphi_{\mathbf{q}}$, representing the rotations in space for points of the unit sphere in $\mathbb{R}^4$. Following up, from the definition of $\varphi_{\mathbf{q}}$, we have $\varphi_{\mathbf{q}} = \varphi_{-\mathbf{q}}$. Geometrically, this fact is obvious because a rotation of an angle θ about the $\mathbf{n}$-axis is the same to a rotation of an angle $2\pi - \theta$ about the $-\mathbf{n}$-axis. Therefore, the transformation $\varphi_{\mathbf{q}}$ is not injective. However, it can be shown, without a lot of difficulty, that

$$\varphi_{\mathbf{q}_1} = \varphi_{\mathbf{q}_2} \Leftrightarrow \mathbf{q}_1 = \pm\mathbf{q}_2.$$

We left that demonstration for the exercises at the end of this chapter.

The above result shows that only the antipode points of the sphere define the same rotation. Therefore, the space $\mathrm{SO}(3)$ is identified naturally with the unit sphere S^3 in $\mathbb{R}^4$ with the antipode points identified. We know this is the real projective space of dimension 3, $\mathbb{RP}^3$. We will not use, however, this association between $\mathrm{SO}(3)$ and $\mathbb{RP}^3$. The important aspect of the above result is that space $\mathrm{SO}(3)$ of the rotations can be represented by unit quaternions. Care should only be taken with antipode points of the sphere representing the same rotation.

An important fact in relation to the representation of rotations in $\mathrm{SO}(3)$ by quaternions, is the product operation is preserved, that is

$$\varphi_{\mathbf{u}_1\mathbf{u}_2} = \varphi_{\mathbf{u}_1}\varphi_{\mathbf{u}_2}.$$

The check is immediate: for every vector $\mathbf{v} \in \mathbb{R}^3$ we have

$$\varphi_{\mathbf{u}_1\mathbf{u}_2}(\mathbf{v}) = \mathbf{u}_1\mathbf{u}_2\hat{\mathbf{v}}(\mathbf{u}_1\mathbf{u}_2)^{-1} = \mathbf{u}_1\mathbf{u}_2\hat{\mathbf{v}}\mathbf{u}_2^{-1}\mathbf{u}_1^{-1} = \mathbf{u}_1\varphi_{\mathbf{u}_2}\,\hat{\mathbf{v}}\,\mathbf{u}_1^{-1} = \varphi_{\mathbf{u}_1}\varphi_{\mathbf{u}_2}(\mathbf{v}).$$

This fact shows we can substitute the product of two rotation matrices by the product of the quaternions representing them. In particular, notice this result answers a question we had in the beginning of the chapter:

> "*If rotations R_1 and R_2 have axes defined by the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, then the axis of the rotation product R_1R_2 is the product of quaternions $\hat{\mathbf{u}}_1\hat{\mathbf{u}}_2$.*"

7.2 Exponential and logarithm

Consider a vector $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} \neq 0$. and let $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ be the pure unit quaternion obtained from $\mathbf{v}$ by normalization. Using the notation $\theta = |\mathbf{v}|$, we have $\mathbf{v} = \theta\hat{\mathbf{v}}$. Substituting the expression $\mathbf{v} = \theta\hat{\mathbf{v}}$ in the Taylor series of the exponential function we obtain,

$$e^{\mathbf{v}} = e^{\theta\hat{\mathbf{v}}} = \sum_{n=0}^{\infty} \frac{(\theta\hat{\mathbf{v}})^n}{n!}.$$

An immediate calculation, taking into account $\hat{\mathbf{v}}^2 = -1$, gives us

$$e^{\mathbf{v}} = e^{\theta\mathbf{v}} = \cos\theta + \hat{\mathbf{v}}\sin\theta = \cos|\mathbf{v}| + \hat{\mathbf{v}}\sin|\mathbf{v}|.$$

We can define the exponential $\exp\colon \mathbb{R}^3 \to S^3$ having

$$\exp(\mathbf{v}) = \begin{cases} \cos|\mathbf{v}| + \sin|\mathbf{v}|\hat{\mathbf{v}}, & \text{se } \mathbf{v} \neq 0; \\ \mathbf{1} = (1,0,0,0) & \text{se } \mathbf{v} = 0, \end{cases}$$

where $\hat{\mathbf{v}} = |\mathbf{v}|/|\mathbf{v}|$. We will also use the notation $\exp(\mathbf{v}) = e^{\mathbf{v}}$. Notice that if $\mathbf{v}$ is a unit quaternion, then $\mathbf{v} = \cos\theta + \sin\theta\hat{\mathbf{u}}$, with $|\mathbf{u}| = 1$. As $|\theta\hat{\mathbf{u}}| = \theta$, we have

$$e^{\theta\mathbf{u}} = \cos\theta + \sin\theta\hat{\mathbf{u}},$$

which is a similar expression to $e^{i\theta} = \cos\theta + i\sin\theta$ for complex numbers.

Now we can define the logarithm function $\log\colon S^3 \to \mathbb{R}^3$, as being the inverse of the exponential. Given $\mathbf{q} \in S^3$, $\mathbf{q} = \cos\theta + \sin\theta\hat{\mathbf{u}}$, $|\hat{\mathbf{u}}| = 1$, we have

$$\log(\mathbf{q}) = \theta\mathbf{u} \in \mathbb{R}^3.$$

It is immediate verifying $e^{\log(\mathbf{q})} = \mathbf{q}$.

It is sometimes useful to have the definition of the exponential and of the logarítham explicitly showing the coordinates; If $\mathbf{u} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in S^3$, we can write

$$\log(w, x, y, z) = \arccos(w)\,\frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\arccos(w)}{\sqrt{x^2 + y^2 + z^2}}(x, y, z).$$

Similarly, if $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have

$$\exp(x, y, z) = \begin{cases} \cos(\sqrt{x^2 + y^2 + z^2}) + \frac{\sin(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}(x, y, z), & \text{se } (x, y, z) \neq 0; \\ (1, 0, 0, 0) & \text{if } (x, y, z) = 0. \end{cases}$$

Now we can define arbitrary potencies of unit quaternions extending in a natural way the result for complex numbers: if $\mathbf{q}$ is a unit quaternion and $t \in \mathbb{R}$, we have

$$q^t = e^{t\log\mathbf{q}} = e^{t\theta\hat{\mathbf{u}}} = \cos(\theta t) + \sin(\theta t)\hat{\mathbf{u}}. \tag{4.12}$$

Observe that the expression $c(t) = \mathbf{q}^t$ (with t varying in the interval $[0, 1]$) defines a curve $c\colon [0, 1] \to S^3$ that ties the point $c(0) = \mathbf{1} = (1, 0, 0, 0)$ (north pole of the sphere S^3) to the point $c(1) = \mathbf{q}$. We will give a geometric interpretation of this curve, and at the same time of the exponential function. For that we needed some definitions.

A *maximum circle* of the sphere S^3 is a circle of radius 1, in other words, a circle contained in S^3 that has maximum radius. A maximum circle is obtained by the intersection of S^3 with a subspace of dimension 2 in $\mathbb{R}^4$. A *geodesic* of S^3 is a maximum circle parameterized for a curve $c\colon [0, 1] \to S^3$ such that the speed $|c'(t)|$ is constant. This implicates that the image of an uniform partition of the parameter space $[0, 1]$ results in an uniform partition of the maximum circle. (The geodesics are the " straight line " of the sphere S^3 when walking between two points on the sphere; The smallest path is always along a geodesic).

Let us return to equation (4.12), from which, geometrically, the curve $c(t) = \mathbf{q}^t$ represents a family of rotations about the same $\mathbf{u}$-axis with given angles by $2\theta t$. In S^3, these rotations represent an arch of maximum circle connecting the quaternion $c(0) = \mathbf{1}$ to the quaternion $c(1) = \mathbf{q}$. We left as an exercise at the end of the chapter the demonstration that

$$c'(t) = \frac{d}{dt}\mathbf{q}^t = q^t \log\mathbf{q}.$$

Then it is easy to verify that the curve $c(t)$ holds the following properties:

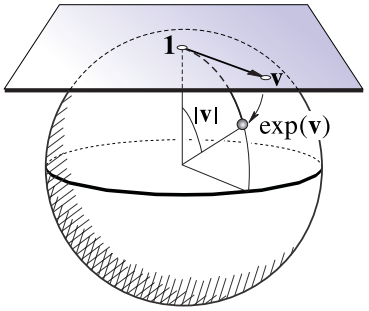


Figure 8. Exponential application.

1. $c'(0) = \theta \mathbf{u}$;
2. $|c'(t)| = |\theta|$;
3. $c''(t) = kc(t)$, where $k < 0$ is a constant.

Condition 3 above shows that $c(t)$, in fact, describes an arch of maximum circle, and condition 2 guarantees us that $c(t)$ is a geodesic. In short, the curve $c(t)$ is a geodesic in S^3, that begins in point $\mathbf{1}$ and ends in point $\mathbf{q}$. (Of course the maximum circle described by $c(t)$ is determined by the intersection of S^3 with the two-dimensional subspace generated by vectors $\mathbf{1}$ and $\mathbf{q}$.)

The previous result provide us with a geometric description of the following exponential application $\exp\colon \mathbb{R}^3 \to S^3$ (see Figure 8): identifying the space $\mathbb{R}^3$ with the tangent plane to the sphere S^3 in the point $\mathbf{1}$ (north pole), $\exp(\mathbf{v})$ is the extreme point of the arch that begins in $\mathbf{1}$, in the direction $\mathbf{v}$ and has length $|\mathbf{v}|$.

In particular, notice the exponential here defined coincides with the exponential defined in Differential Geometry for any surface of $\mathbb{R}^n$ with the induced metric.

Exponential parameterization of SO(3)

The result of the previous section shows that the exponential here defined naturally extends the exponential $\exp\colon \mathbb{R} \to S^1$ that we defined in the beginning of the chapter. In this case, the exponential allowed a parameterization of the space of rotations of the plane for a subset of $\mathbb{R}$. In ours case also, the exponential $\exp\colon \mathbb{R}^3 \to S^3$ naturally defines a parameterization of the space of rotations $\mathrm{SO}(3)$.

Given a vector $\mathbf{v} \in \mathbb{R}^3$, we have $\exp(\mathbf{v}) = \cos|\mathbf{v}| + \sin|\mathbf{v}|\hat{\mathbf{u}}$, where $\mathbf{u} = \mathbf{u}/|\mathbf{u}|$. Therefore, $\exp(\mathbf{v})$ is a unit quaternion representing a rotation about the $\mathbf{u}$-axis (this is the same axis of $\mathbf{v}$), of an angle $2|\mathbf{v}|$. That is, the exponential is a parameterization capturing the essence of parameterizing the space of rotations in the $\{\text{axis} + \text{angle}\}$ space, as we

promised previously: the axis is modeled by a vector, and the angle is its length (less of factor 2).

As we are parameterizing $\mathrm{SO}(3)$ by $\mathbb{R}^3$, of course, the exponential holds singularities. Using the geometric interpretation of the exponential it is easy to conclude that the spheres of radius $k\pi$, $k = 1, 2, \ldots$ are singularities regions of this parameterization. They are the regions of $\mathbb{R}^3$ mapped in the north pole or in the south pole of the sphere S^3. These singularities can be avoided in practice: when $\mathbf{v}$ approaches the sphere of radius π, we have a rotation of an angle $\theta = 2|\mathbf{v}|$ close to 2π, we then change for a rotation of an angle $2\pi - \theta$ about the $-\mathbf{v}$-axis that is equivalent and far away from the singularity.

Observing the definition of exponential parameterization attentively, we notice that we can have a problem of numerical instability in the calculation of the unit vector $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$, when $v \to 0$. This problem can be avoided. In fact, we have

$$\begin{aligned} e^{\mathbf{v}} &= \cos|\mathbf{v}| + \sin|\mathbf{v}|\hat{\mathbf{u}} \\ &= \cos|\mathbf{v}| + \sin|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \cos|\mathbf{v}| + \frac{\sin|\mathbf{v}|}{|\mathbf{v}|}\mathbf{v}. \end{aligned}$$

Now we only need to observe that $\frac{\sin|\mathbf{v}|}{|\mathbf{v}|} \to 1$ when $|\mathbf{v}| \to 0$. In reality, for implementation purpose, we can substitute this function by an approximation given by its Taylor series:

$$\frac{\sin t}{t} = 1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \cdots + \frac{t^{2(n-1)}}{(2n-1)!} + \cdots$$

7.3 Interpolating quaternions

In this section we will cover the topic of interpolating rotations using the representation by quaternions. We will just study the case for two rotations. Let us consider two rotations $\varphi_{\mathbf{u}}$ and $\varphi_{\mathbf{v}}$ of angles $\theta_{\mathbf{u}}$ and $\theta_{\mathbf{v}}$ about unit axes $\mathbf{u}$ and $\mathbf{v}$, respectively. These two rotations are represented by unit quaternions as indicated below:

$$\begin{aligned} \varphi_{\mathbf{u}} &\longleftrightarrow \mathbf{p} = e^{\theta_{\mathbf{u}}\mathbf{u}} = \cos\theta_{\mathbf{u}} + \sin\theta_{\mathbf{u}}\hat{\mathbf{u}} \\ \varphi_{\mathbf{v}} &\longleftrightarrow \mathbf{q} = e^{\theta_v\mathbf{v}} = \cos\theta_{\mathbf{v}} + \sin\theta_{\mathbf{v}}\hat{\mathbf{v}} \end{aligned}$$

We then have

$$\varphi_u(\mathbf{x}) = \mathbf{p}\hat{\mathbf{x}}\mathbf{p}^* \quad \text{e} \quad \varphi_v(\mathbf{x}) = \mathbf{q}\hat{\mathbf{x}}\mathbf{q}^*.$$

Quaternions $\mathbf{p}$ and $\mathbf{q}$ define an arch of maximum circle in the sphere S^3 (see Figure 9(a)). Due to the representation of rotations by unit quaternions, an interpolation between the rotations φ_u and φ_v is simply a curve $g\colon [0,1] \to S^3$ in the sphere S^3 such that $g(0) = \mathbf{p}$ and $g(1) = \mathbf{q}$. That is, the interpolation of rotations is reduced to a problem of interpolating points in S^3.

Linear Interpolation

As quaternions hold the usual linear structure of $\mathbb{R}^4$, we can do a linear interpolation between $\mathbf{p}$ and $\mathbf{q}$. More precisely, we define the quaternion

$$\mathbf{a}(t) = (1-t)\mathbf{p} + t\mathbf{q},$$

and the interpolated rotation is given by $\varphi_{\mathbf{a}(t)}$ where

$$\varphi_{\mathbf{a}(t)}(\mathbf{x}) = \mathbf{a}(t)\hat{\mathbf{x}}\mathbf{a}(t)^*.$$

The problem of this method is that $\mathbf{a}(t)$ are not unit quaternions. In fact, geometrically they form a " rope " of the maximum circle of the unit sphere $S^3 \subset \mathbb{R}^4$ containing the quaternions $\mathbf{p}$ and $\mathbf{q}$ (see Figure 9(b)).

We can bypass this problem by doing a *spherical radial interpolation*, consisting of taking the radial projection of $\mathbf{a}(t)$ on the unit sphere, as shown in Figure 9(b). That is,

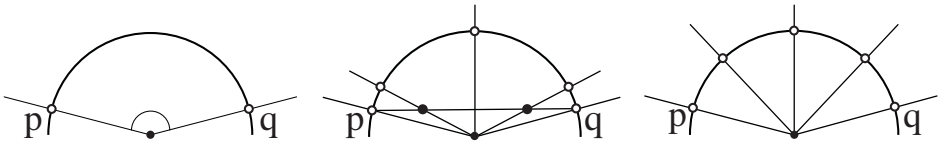
$$\overline{\mathbf{a}}(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|} = \frac{(1-t)\mathbf{p} + t\mathbf{q}}{|(1-t)\mathbf{p} + t\mathbf{q}|}.$$

This method works; However, we do not have an uniform sampling in the angles even when the sampling is uniform in time. (That is, the projection is a maximum circle but it is not a geodesic.) This fact can be observed in Figure 9(c). In other words, the angular speed is variable (it accelerates and decelerates.).

Geodesic interpolation

Our goal now is to correct the problem of spherical radial interpolation and to obtain a interpolation between two quaternions in which the interpolating curve has constant angular speed, that is, an uniform sampling in time generates an uniform sampling in the angles. As we know already, it is enough to do the interpolation along a geodesic of the sphere S^3 tieing quaternions $\mathbf{p}$ to $\mathbf{q}$. This method is called *geodesic interpolation*.

Previously, we saw that curve $c(t) = \mathbf{b}^t$ is a geodesic tying the north pole $\mathbf{1}$ of the sphere to quaternion $\mathbf{b}$. This provides a geodesic interpolation between rotations $\varphi_{\mathbf{1}}$ and $\varphi_{\mathbf{b}}$.



(a) (b) (c)

Figure 9. Quaternion interpolation.

How to obtain a geodesic tieing $\mathbf{p}$ to $\mathbf{q}$? A simple calculation shows the curve

$$g(t) = (\mathbf{qp})^t\mathbf{p}, \quad t \in [0,1], \tag{4.13}$$

satisfies $g(0) = \mathbf{p}$ and $g(1) = \mathbf{q}$. It remains to show $g(t)$ is a geodesic. For this, observe that $g(t) = R_{\mathbf{p}}(c(t))$ where $c(t) = (\mathbf{qp})^t$ and $R_{\mathbf{p}}$ is the transformation of S^3 defined by $R_{\mathbf{p}}(\hat{\mathbf{v}}) = \hat{\mathbf{v}}\mathbf{p}$. Now we know $c(t)$ is a geodesic; On the other hand, the transformation $R_{\mathbf{p}}$ is orthogonal, therefore preserving geodesics for being an isometry of the sphere. (In more geometric terms: a geodesic is a maximum circle and sphere rotations preserve maximum circles.)

In conclusion, equation (4.13) defines a geodesic interpolation between quaternions $\mathbf{p}$ and $\mathbf{q}$. This interpolation method is called *spherical linear interpolation* or, briefly, slerp.

Next, we will obtain an expression of geodesic interpolation which is very common in the literature; It expresses the interpolation parameter in terms of angles in the maximum circle tieing $\mathbf{p}$ to $\mathbf{q}$. Let θ be the angle between quaternions $\mathbf{p}$ and $\mathbf{q}$, that is, $\cos\theta = \langle \mathbf{p}, \mathbf{q} \rangle$ (Figure 10(a)).

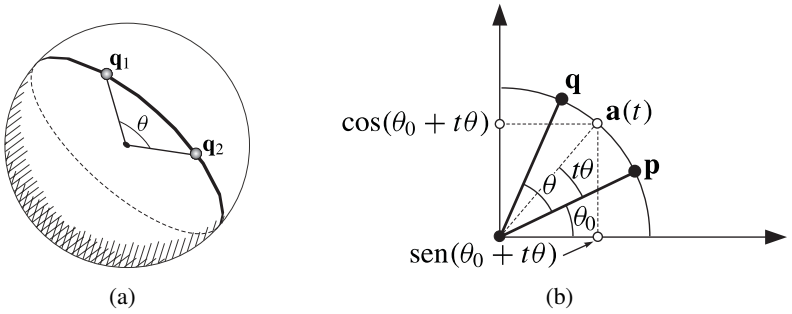


Figure 10. Spherical linear Interpolation.

As we will work with angles of the maximum circle, our problem is reduced to a planar one. (That is, we work in the plane of the maximum circle, defined by $\mathbf{p}$, $\mathbf{q}$ and the origin). Observing Figure 10(b), we have:

$$\begin{aligned} \mathbf{p} &= (\cos(\theta_0), \sin(\theta_0)), \\ \mathbf{q} &= (\cos(\theta_0 + \theta), \sin(\theta_0 + \theta)). \end{aligned}$$

Therefore, in terms of angles, the interpolation between $\mathbf{p}$ and $\mathbf{q}$ is given by

$$\mathbf{g}(t) = (\cos(\theta_0 + t\theta), \sin(\theta_0 + t\theta)).$$

Now we should write this expression as function of quaternions p and q. That is achieved with a simple calculation, using trigonometry (we will indicate a vector (x, y) in the plane by a column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, due to space contraints):

$$\begin{aligned}
\mathbf{g}(t) &= \begin{pmatrix} \cos(\theta_0 + t\theta) \\ \sin(\theta_0 + t\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_0)\cos(t\theta) - \sin(\theta_0)\sin(t\theta) \\ \sin(\theta_0)\cos(t\theta) + \cos(\theta_0)\sin(t\theta) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\cos(\theta_0)[\sin(\theta)\cos(t\theta) - \cos(\theta)\sin(t\theta)] + [\cos(\theta_0)\cos(\theta) - \sin(\theta_0)\sin(\theta)]\sin(t\theta)}{\sin(\theta)} \\ \frac{\sin(\theta_0)[\sin(\theta)\cos(t\theta) - \cos(\theta)\sin(t\theta)] + [\sin(\theta_0)\cos(t\theta) + \cos(\theta_0)\sin(\theta)]\sin(t\theta)}{\sin(\theta)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\cos(\theta_0)\sin((1-t)\theta) + \cos(\theta_0+\theta)\sin(t\theta)}{\sin(\theta)} \\ \frac{\sin(\theta_0)\sin((1-t)\theta) + \sin(\theta_0+\theta)\sin(t\theta)}{\sin(\theta)} \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} \frac{\sin((1-t)\theta)}{\sin(\theta)} + \begin{pmatrix} \cos(\theta_0+\theta) \\ \sin(\theta_0+\theta) \end{pmatrix} \frac{\sin t\theta}{\sin(\theta)} \\
&= \mathbf{p}\frac{\sin((1-t)\theta)}{\sin\theta} + \mathbf{q}\frac{\sin(t\theta)}{\sin\theta}.
\end{aligned}$$

In short, we have

$$\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}\frac{\sin((1-t)\theta)}{\sin\theta} + \mathbf{q}\frac{\sin(t\theta)}{\sin\theta}. \tag{4.14}$$

7.4 Some afterthoughts

The representation of rotations by quaternions brings many advantages. Among them we can enumerate:

1. Smaller storage space (a quaternion has four components and a matrix holds nine, or 16 in homogeneous coordinates).

2. We have a global parameterization, thus facilitating the interpolation of rotations (we interpolated in the unit sphere $S^3 \subset \mathbb{R}^4$).

3. The interpolation of rotations is reduced to a problem of interpolating points in the unit sphere (unit quaternions).

4. The exponential parameterization translates the natural way we think about rotation: axis+angle.

5. The representation is intrinsic (i.e. it does not depend on coordinates)

6. Quaternion product can be implemented in a more efficient way than matrix product.

On this last item, notice that multiplication of two quaternions has 16 products, while the multiplication of matrices has 27 products. In reality, even more surprisingly, the multiplication of two quaternions can be obtained with only 8 products (see exercises).

A problem with using quaternions is that several existent geometry libraries are in general designed to work with matrices, and in particular with Euler's angles. Therefore, it is convenient to have the expressions to convert among the several representations (i.e. matrix, Euler's angles, quaternions, exponential parameterizatione). We will cover this subject in the next section.

8 Converting among representations

In this section we will study two types of conversions among representations: from quaternions to matrix, and from Euler's angles to quaternions. We already saw previously the conversion from Euler's angles to matrices.

8.1 Quaternions and matrices

In this section we will find the rotation matrix represented by

$$\varphi_{\mathbf{q}}(\mathbf{v}) = \mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{-1},$$

where $\mathbf{q}$ is a unit quaternion.

Before, let us draw a parallel with complex numbers. There exists a relation between the product of complex numbers and linear transformations. Let us associate, to each complex number $z = a + bi$, a linear transformation $L_z \colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$L_z(x, y) = z(x + iy) = (a + bi)(x + iy) = (ax - by, bx + ay).$$

The transformation matrix of L_z is given by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

which can be verified with a direct calculation. We will extend this result for quaternions.

Initially, we observed that when a quaternion $\mathbf{q} = w\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ represents a vector of $\mathbb{R}^3$ in homogeneous coordinates, the infinite coordinate is given by variable w. In Chapter 2 (Geometry) we took the coordinate of the point in the infinite as being the last coordinate. In this case, the vector corresponding to quaternion $\mathbf{q}$ is given by (x, y, z, w).

Let $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)$ and $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)$ be two quaternions. From the bilinearity of quaternion product, we know the product $\mathbf{q}_1\mathbf{q}_2$ is linear in $\mathbf{q}_1$ and $\mathbf{q}_2$. Therefore we have two linear transformations: $L_{\mathbf{q}_1} : \mathbb{R}^4 \to \mathbb{R}^4$, and $R_{\mathbf{q}_1} : \mathbb{R}^4 \to \mathbb{R}^4$, defined respectively, for the product to the left and right of quaternion $\mathbf{q}_1$:

$$L_{\mathbf{q}_1}(\mathbf{q}) = \mathbf{q}_1\mathbf{q}, \quad \text{e} \quad R_{\mathbf{q}_1}(\mathbf{q}) = \mathbf{q}\mathbf{q}_1.$$

A direct calculation shows the matrices of these linear transformations are given by

$$L_{\mathbf{q}_1}(\mathbf{q}) = \begin{pmatrix} w_1 & -z_1 & y_1 & x_1 \\ z_1 & w_1 & -x_1 & y_1 \\ -y_1 & x_1 & w_1 & z_1 \\ -x_1 & -y_1 & -z_1 & w_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{4.15}$$

and

$$R_{\mathbf{q}_1}(q) = \begin{pmatrix} w_1 & z_1 & -y_1 & x_1 \\ -z_1 & w_1 & x_1 & y_1 \\ y_1 & -x_1 & w_1 & z_1 \\ -x_1 & -y_1 & -z_1 & w_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{4.16}$$

We can use any one of these two transformations to represent the product of two quaternions using linear transformations. We observed that this association is analog to the association between complex numbers and matrices we presented in the beginning of this section. Here we have two possibilities because the product of two quaternions it is not commutative. (Notice that if $\mathbf{q}_1$ is unitary, then transformations $L_{\mathbf{q}_1}$ and $R_{\mathbf{q}_1}$ are orthogonal.)

It is important to highlight that this relation between quaternions and linear transformations preserves the product and sum operations in each one of these spaces. More precisely, quaternion $\mathbf{q}_1(\mathbf{q}_2+\mathbf{q}_3)$ corresponds to matrix $L_{\mathbf{q}_1}(L_{\mathbf{q}_2}+L_{\mathbf{q}_3})$, or $R_{\mathbf{q}_1}(R_{\mathbf{q}_2}+R_{\mathbf{q}_3})$.

As quaternions product is associative, we have

$$(\mathbf{q}_1\mathbf{p})\mathbf{q}_2 = \mathbf{q}_1(\mathbf{p}\mathbf{q}_2).$$

Using transformations L and R this expression can be written in the form

$$R_{\mathbf{q}_2}L_{\mathbf{q}_1}p = L_{\mathbf{q}_1}R_{\mathbf{q}_2}\mathbf{p}.$$

This relation is very important from the computational point of view. In fact, in a quaternion product

$$\mathbf{q}_1\mathbf{q}_2 \dots \mathbf{q}_{i-1}\mathbf{q}_i\mathbf{q}_{i+1} \dots \mathbf{q}_N, \tag{4.17}$$

if only quaternion q_i varies, we can combine the products $\mathbf{q}_1\mathbf{q}_2 \dots \mathbf{q}_{i-1}$ in a matrix $L_{\mathbf{q}_1\mathbf{q}_2\dots\mathbf{q}_{i-1}}$, and the product $\mathbf{q}_{i+1} \dots \mathbf{q}_N$ in a matrix $R_{\mathbf{q}_{i+1}\dots\mathbf{q}_N}$, and then write the product of (4.17) in the form

$$L_{\mathbf{q}_1\mathbf{q}_2\dots\mathbf{q}_{i-1}}R_{\mathbf{q}_{i+1}\dots\mathbf{q}_N}\mathbf{q}_i.$$

We will now find the matrix in $\mathrm{SO}(3)$ corresponding to the rotation defined by $\varphi_{\mathbf{q}}(\mathbf{v}) = \mathbf{q}\hat{\mathbf{v}}\mathbf{q}^{-1}$. As $\mathbf{q}$ is unitary, we have $\mathbf{q}^{-1} = \mathbf{q}^{\star}$, and therefore it is proceeded that

$$\varphi_{\mathbf{q}}(\mathbf{v}) = L(\mathbf{q})R(\mathbf{q}^{\star})(\hat{\mathbf{v}}). \tag{4.18}$$

If $\mathbf{q} = (x, y, z, w)$, then $\mathbf{q}^{\star} = (-x, -y, -z, w)$. Using the matrix of $L_{\mathbf{q}_1}$ in (4.15) with $\mathbf{q}_1 = \mathbf{q}$ and the matrix $R_{\mathbf{q}_1}$ in (4.16) with $\mathbf{q}_1 = \mathbf{q}^{\star}$, and substituting in (4.18), we obtain

$$L(\mathbf{q})R(\mathbf{q}^{*}) = \begin{pmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{pmatrix} \begin{pmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{pmatrix}.$$

Doing the product, we obtain the matrix

$$\begin{pmatrix} w^2+x^2-y^2-z^2 & 2xy-2wz & 2xz+2wy & 0 \\ 2xy+2wz & w^2-x^2+y^2-z^2 & 2yz-2wx & 0 \\ 2xz-2wy & 2yz+2wx & w^2-x^2-y^2+z^2 & 0 \\ 0 & 0 & 0 & |\mathbf{q}|^2 \end{pmatrix}. \tag{4.19}$$

(See exercise 3 for another form of writing this matrix.) As the quaternion $\mathbf{q}$ is unitary, we have

$$|\mathbf{q}|^2 = w^2 + x^2 + y^2 + z^2 = 1.$$

In Cartesian coordinates the above matrix can be written in the form

$$\begin{pmatrix} w^2+x^2-y^2-z^2 & 2xy+2wz & 2xz-2wy \\ 2xy-2wz & w^2-x^2+y^2-z^2 & 2yz+2wx \\ 2xz+2wy & 2yz-2wx & w^2-x^2-y^2+z^2 \end{pmatrix}$$

as well as,

$$2\begin{pmatrix} \frac{1}{2}-y^2-z^2 & xy+wz & xz-wy \\ xy-wz & \frac{1}{2}-x^2-z^2 & yz+wx \\ xz+wy & yz-wx & \frac{1}{2}-x^2-y^2 \end{pmatrix}.$$

It is also important to solve the problem of finding the quaternion associated with matrix L or R. We left this problem for the exercises at the end of the chapter.

8.2 Quaternions and Euler's angles

Another important relation is to obtain the quaternion associated with a rotation using Euler's angles α β and γ. We have three rotations $R_x(\alpha)$, $R_y(\beta)$, $R_z(\gamma)$ of angles α, β, γ about axes $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. Each one of these rotations is represented by $\varphi_{\mathbf{q}_x}$, $\varphi_{\mathbf{q}_y}$, and $\varphi_{\mathbf{q}_z}$, where

$$\mathbf{q}_x = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}(1,0,0),$$
$$\mathbf{q}_y = \cos\frac{\beta}{2} + \sin\frac{\beta}{2}(0,1,0),$$
$$\mathbf{q}_z = \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}(0,0,1).$$

The rotation in space is given by $R_z(\gamma)R_y(\beta)R_x(\alpha)$, and it will be represented by the quaternion $\mathbf{q} = \mathbf{q}_z\mathbf{q}_y\mathbf{q}_x$, obtained from the product of the three quaternions. Supposing

$$\mathbf{q} = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

and doing the calculations, we obtain

$$w = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$
$$x = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$
$$y = \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$$
$$z = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

9 Comments and References

Hamilton's goal on searching for quaternions consisted on discovering a multiplication structure, similar to the one of complex numbers, for the $\mathbb{R}^3$ and that could be used to represent rotations in space. He ended up discovering such structure for the $\mathbb{R}^4$. Today it is known this type of structure exists only in $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$.

Quaternions were introduced in Computer Graphics in the article (Shoemake, 1985). They were already used in the area of Robotics.

For more information about orthogonal transformations of $\mathbb{R}^n$, and in particular, the demonstration of Euler's Theorem in its general form, we suggest (Lima, 1999).

9.1 Additional Topics

In this chapter we studied the special group $\mathrm{SO}(3)$ of rotations in space which consists in just a part of the representation of rigid motions in space. The complete group, has dimension 6 (three degrees of freedom for rotations and three for translations), and is called *special Euclidean group*, denoted by $\mathrm{SE}(3)$. An appropriate representation of this group involves the screw theory and the study of the Lie Groups of matrices. Complete material on these topics can be found in a good textbook in the area of Robotics.

Quaternions just represent the tip of the iceberg in the study of intrinsic operations among geometric objects, a topic covered in the area of Geometric Algebra.

An important topic that we did not cover is the problem of interpolating the n rotations represented by quaternions. From what we covered, this problem is equivalent to the study of interpolation curves (e.g. Bezier, Splines) in the sphere S^3. Several works exist about it on the literature. This subject is also very important for the area of GIS (Geographic Information Systems). The interpolation of rotations using the structure of Lie Groups of matrices is an interesting topic. (Alexa, 2002), for instance, describes the interpolation of linear transformations using results from Lie Groups of matrices.

10 Exercises

1. Demonstrate Chasles' Theorems: " Every positive rigid motion in $\mathbb{R}^3$ can be obtained by a rotation about a r-axis, followed by a translation along r " (this type of motion is called *screw*).

2. Describe a method to obtain the quaternion associated with matrix L or R (as defined in this chapter), in homogeneous coordinates.

3. If $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation, show that $R(u \wedge v) = R(u) \wedge R(v)$.

4. Show there exists no multiplicative structure in $\mathbb{R}^3$ among the vectors, similar to quaternions (or to complex numbers). (Suggestion: use the transformation $L_{\mathbf{q}}$ or $R_{\mathbf{q}}$ defined in the chapter, including the fact that a linear transformation in $\mathbb{R}^3$ has 1 real eingenvalue.)

5. If (r_{ij}) is a matrix of order 3 representing a rotation in $\mathbb{R}^3$, show that the Euler's angles of the parameterization given by $R_z(\alpha)R_y(\beta)R_z(\gamma)$ are determined by:

$$\begin{aligned}
\alpha &= \operatorname{arctg2}(\sqrt{r_{31}^2 + r_{32}^2}, r_{33});\\
\beta &= \operatorname{arctg2}(\frac{r_{23}}{\sin\beta}, \frac{r_{13}}{\sin\beta});\\
\gamma &= \operatorname{arctg2}(\frac{r_{32}}{\sin\beta}, -\frac{r_{31}}{\sin\beta}),
\end{aligned}$$

where $\operatorname{arctg2}(x, y)$ is the function $\operatorname{arctg}(x/y)$ using the signal of x and y to determine the quadrant of the resulting angle.

6. Consider the following problem: *" to implement an interactive interface to rotate an object of $\mathbb{R}^3$ about an axis, just using the mouse as input device."* Two solutions exist in the literature for this problem: the *Metaball* and the *virtual sphere*.

a) Describe the model of these two solutions;

b) Compare the solutions and discuss the advantages and disadvantages of each one.

7. Show if a quaternion $\mathbf{q}$ commutes with every pure quaternion, then $\mathbf{q}$ is real. Use this fact to show that

$$\varphi_{\mathbf{q}_1} = \varphi_{\mathbf{q}_2} \Leftrightarrow \mathbf{q}_1 = \pm\mathbf{q}_2,$$

where $\varphi_{\mathbf{q}}$ is the representation of a rotation defined in (4.9).

8. If $\mathbf{p} = \cos\theta + \hat{\mathbf{v}}\sin\theta$ is a unit quaternion, and $t \in \mathbb{R}$, show that $\mathbf{q}\mathbf{p}^t\mathbf{q}^* = (\mathbf{q}\mathbf{p}\mathbf{q}^*)^t$.

9. If $\mathbf{q}$ is a unit quaternion, and $a, b \in \mathbb{R}$, show that:

$$\mathbf{q}^a\mathbf{q}^b = \mathbf{q}^{a+b} \quad \text{e} \quad (\mathbf{q}^a)^b = \mathbf{q}^{ab}.$$

10. The calculations below show that the product of quaternions is commutative:

$$\begin{aligned} pq &= \exp(\log(pq)) \\ &= \exp[\log(p) + \log(q)] \\ &= \exp[\log(q) + \log(p)] \\ &= \exp(\log(q))\ \exp(\log(p)) = qp. \end{aligned}$$

Where is the error?

11. Show that, in the case in which quaternion $\mathbf{q}$ is not unitary, the matrix (4.19) of the transformation $\varphi_{\mathbf{q}}$ is given by

$$\frac{2}{|\mathbf{q}|^2}\begin{pmatrix} \frac{|\mathbf{q}|^2}{2} - y^2 - z^2 & xy + wz & xz - wy & 0 \\ xy - wz & \frac{|\mathbf{q}|^2}{2} - x^2 - z^2 & yz + wx & 0 \\ xz + wy & yz - wx & \frac{|\mathbf{q}|^2}{2} - x^2 - y^2 & 0 \\ 0 & 0 & 0 & \frac{|\mathbf{q}|^2}{2} \end{pmatrix}.$$

12. Show that

$$e^{\theta\hat{\mathbf{u}}} = \cos\theta + \hat{\mathbf{u}}\sin\theta,$$

substituting the expression of $\theta\hat{\mathbf{u}}$ in the series of potencies of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n}.$$

13. Let

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

be a rotation matrix in $\mathbb{R}^3$. Show that the rotation axis is the vector

$$v = \frac{1}{\sin\theta}(r_{32} - r_{23}.r_{13} - r_{31}, r_{21} - r_{12}).$$

If the angle of rotation θ is different from π and 0, then

$$\theta = \arccos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right).$$

What happens if the rotation angle is very small? Describe a robust method to find the rotation axis (your method should include the cases for $\theta = 0$ and $\theta = \pi$).

14. Using the previous exercise, show that the matrix

$$R_y(90^\circ)R_z(90^\circ) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

represents a rotation of 120° about the axis of the $\mathbb{R}^3$ space defined by the unit vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

15. If $\mathbf{q}(t)$ is a curve in the unit sphere S^3, then $\mathbf{q}'(t) = \mathbf{v}\hat{(}t)\mathbf{q}(t)$, where $\hat{\mathbf{v}}$ represents a pure quaternion associated with the vector $\mathbf{v} \in \mathbb{R}^3$. (Suggestion: Do $\mathbf{q}(t+s) = \mathbf{q}(t+s)\mathbf{q}(t)\mathbf{q}(t)^{-1}$, and use $\mathbf{q}'(t) = d\mathbf{q}(t+s)/ds$ in the point $s = 0$.)

16. If $\mathbf{q}$ is a unit quaternion, show that

$$\frac{d}{dt}\mathbf{q}^t = \mathbf{q}^t \log(\mathbf{q}).$$

17. Show the veracity of the equalities below:

a) $\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}(\mathbf{p}^*\mathbf{q})^t$;

b) $\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = (\mathbf{p}\mathbf{q}^*)^{1-t}\mathbf{q}$;

c) $\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = (\mathbf{q}\mathbf{p}^*)^t\mathbf{p}$;

d) $\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = \mathbf{q}(\mathbf{q}^*\mathbf{p})^{1-t}\mathbf{q}$.

Then conclude that $\operatorname{slerp}(\mathbf{p}, \mathbf{q}, t) = \operatorname{slerp}(\mathbf{q}, \mathbf{p}, 1-t)$.

18. If $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, $\mathbf{v} \neq 0$, show the Jacobian matrix $d\exp_{\mathbf{v}}$ of the exponential function in the point $\mathbf{v}$ is given by

$$\begin{pmatrix} -\frac{s}{|\mathbf{v}|}x & -\frac{s}{|\mathbf{v}|}y & -\frac{s}{\mathbf{v}}z \\ \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)x^2 + \frac{s}{|\mathbf{v}|} & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)xy & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)xz \\ \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)xy & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)y^2 + \frac{s}{|\mathbf{v}|} & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)yz \\ \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)xz & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)yz & \left(\frac{c}{|\mathbf{v}|^2} - \frac{s}{|\mathbf{v}|^3}\right)z^2 + \frac{s}{|\mathbf{v}|} \end{pmatrix},$$

where $s = \sin|\mathbf{v}|$ and $c = \cos|\mathbf{v}|$.

19. Show the Jacobian matrix of the exponential function at the origin $\mathbf{v} = (0,0,0) \in \mathbb{R}^3$, is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Suggestion: use the expression of the previous exercise for $\mathbf{v} \neq 0$, together with the definition of the derivative as a limit.)

20. This exercise shows that the multiplication of two quaternions can be made with only 8 products instead of 16 products as would be expected. Given two quaternios $q_1 = (x_1, y_1, z_1, w_1)$ and $q_2 = (x_2, y_2, z_2, w_2)$, do

$$\begin{aligned} a_1 &= (x_1 - y_1)(y_2 - z_2) & a_5 &= (z_1 - x_1)(x_2 - y_2) \\ a_2 &= (w_1 + x_1)(w_2 + x_2) & a_6 &= (z_1 + x_1)(x_2 + y_2) \\ a_3 &= (w_1 - x_1)(y_2 + z_2) & a_7 &= (w_1 + y_1)(w_2 - z_2) \\ a_4 &= (z_1 + y_1)(w_2 - x_2) & a_8 &= (w_1 - y_1)(w_2 + z_2). \end{aligned}$$

Define $s = a_6 + a_7 + a_8$ and $t = (a_5 + s)/2$. Show that

$$q_1 q_2 = (a_2 + t - s)\mathbf{1} + (a_2 + t - s)\mathbf{i} + (a_3 + t - a_8)\mathbf{j} + (a_4 + t - a_7)\mathbf{k}.$$

Bibliography

Alexa, Marc. 2002. Linear Combination of Transformations. *ACM Transactions on Graphics*, **21**(3), 380–387.

Lima, Elon L. 1999. *Álgebra Linear*. (Terceira edição). Instituto de Matemática Pura e Aplicada, Rio de Janeiro: Coleção Matemática Universitária.

Shoemake, K. 1985 (July). Animating Rotation with Quaternion Curve. *Pages 245–254 of: Computer Graphics (SIGGRAPH '85 Proceedings)*, vol. 19.

Índice

Euler
    angles of, 7
Euler's angles, 7
    and quaternions, 29

geodesic, 21

interpolation
    geodesic, 24
    radial spherical, 24

maximum circle, 21

parameterization
    exponential, 22
    place, 10

quaternions, 4, 14
    and Euler's angles, 29
    and matrices, 27
    and rotations, 16
    conjugated, 14

rotation
    and quaternions, 16
    Euler's angles, 7
    in the space, 4
    matrix of Euler, 9
    on the plane, 2
    quaternions, 4
    representation by quaternions, 14
    straight line nodal, 7

yaw, pitch and roll, 8
slerp, 24, 26

yaw, pitch and roll, 8