Definition of a Geometry

- Synthetic Model
  - Axioms $\rightarrow$ Theorems
- Analitic Model
  - Coordinates

Euclidean Space

- Vector Space

  $\mathbf{R}^n$, $n = 2, 3$

- Coordinates

  $x = (x_1, \ldots, x_n)$

- Inner Product

  $< x, y >= \sum_i x_i y_i$

- Lenght (Norm)

  $||x|| = \sqrt{<} x, x >$

- Tools
  - Linear Algebra

Linear Operators

$T : \mathbf{R}^n \to \mathbf{R}^n$

- Definition

  $T(x + y) = T(x) + T(y)$

  $T(\lambda x) = \lambda T(x)$

- Invertible

- Linear Invariance
  - Subspaces $\to$ Subspaces

    (lines $\to$ lines)

    (origin $\to$ origin)

- Example 2D:

  $x' = T(x)$

  $(x'_1, x'_2) = T[(x_1, x_2)]$

  $(x'_1, x'_2) = (ax_1 + cx_2, bx_1 + dx_2)$

Matrix Interpretation

M − Matrix $n \times n$

- Isomorphism
  - Algebra of Linear Operators
  - Algebra of Matrices

$T \leftrightarrow M$

$$x' = T(x)$$

$$v \to vM$$

$$(x_1 x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$= \begin{pmatrix} ax_1 + cx_2 & bx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} x'_1 & x'_2 \end{pmatrix}$$

Isometries

- Metric Preserving

$$||Tx|| = ||x||$$

- Invertible
- Orthonormal Matrix
- Transformations
  - Rotation
  - Miror

Linear Transformations

- Scaling
- Rotation
- Shear
- Mirror

Affine Transformations

$$A(x) = M(x) + t$$

- Preserve
  - Ratios
  - Proportions
- Transformations
  - Linear Transformations
  - Translations

Transforming Elements

- Points
- Vectors

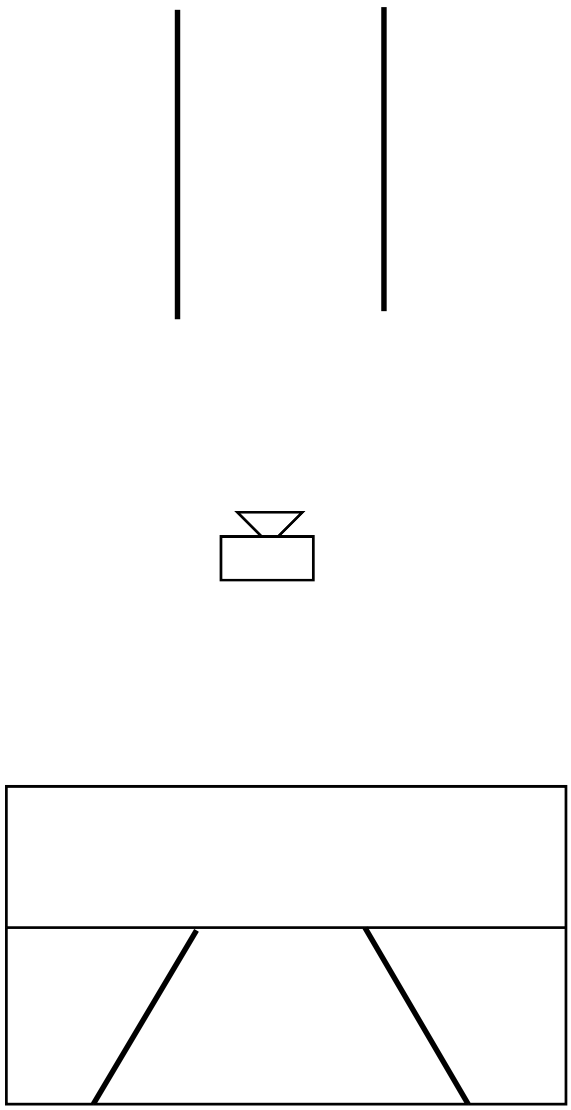
Best Geometry for CG ?

- Synthetic Model

  given point p and line l

- Euclidean
  - only one parallel
- Hyperbolic
  - many parallels
- Projective
  - no parallel

Example: Photography



Euclidean Geometry

- Concepts
  - Congruency
  - Similarity
- Transformations
  - Rigid Motions (Isometry)
  - Uniform Scaling (angle)
- Invariant
  - Angles
  - Parallelism

Projective Geometry

- Concepts
  - Projections
- Transformations
  - Linear
  - Translations
  - Perspective
- Invariant
  - Linear Structures

Projective Plane

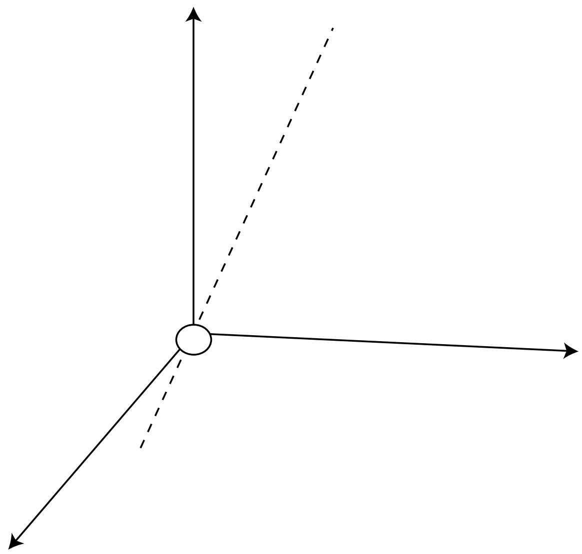
- Point in RP^2

  $$p = (\lambda x_1, \lambda x_2, \lambda x_3) : \lambda \neq 0$$

- Equivalence Relation

  $$p = (x_1, x_2, x_3) \equiv \lambda p$$

$$RP^2 := \mathbf{R}^3 - \{(0,0,0)\}$$



Decomposition of RP^2

- Affine Plane Immersed in RP^2

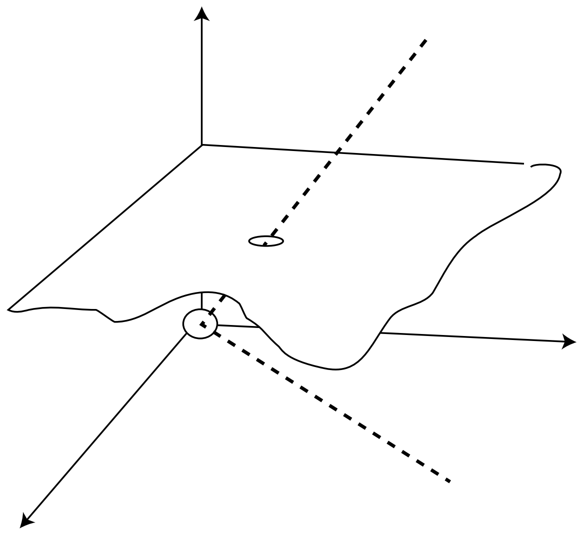
  $\pi : x_3 = 1$

- Normalization

  $p = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)$, $x_3 \neq 0$

- Partition of RP^2

  $RP^2 = \{(x_1, x_2, x_3), x_3 \neq 0\} \cup \{(x_1, x_2, 0)\}$



Homogeneous Coordinates

$p = \lambda p = (\lambda x_1, \lambda x_2, \lambda x_3)$

- Affine Points - $p \in \pi$

  $p_a = (x_1, x_2, 1)$

  $x_3 \neq 0$

  $\lambda = \frac{1}{x_3}$

- Ideal Points - $p \notin \pi$

  $p_i = (x_1, x_2, 0)$

  $x_3 = 0$

  $\lambda = 1$

- Homogeneous Point

  $p = (x_1, x_2, x_3)$

  – no distintion

Projective Transformations

- Linear Operator in $\mathbf{R}^3$

  $T : \mathbf{R}^3 \to \mathbf{R}^3$

- T given by M

  M, matrix 3×3

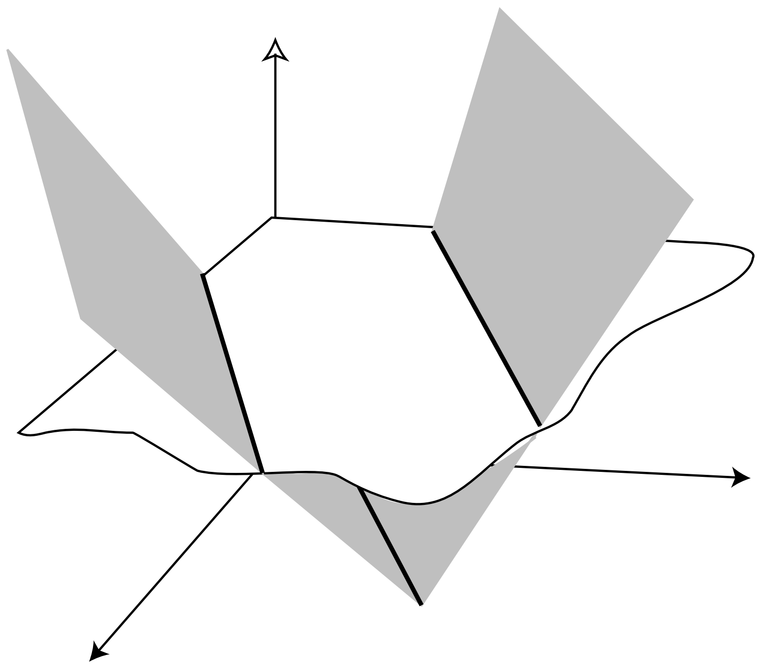
- Projective Transformation Induced by T

  $T(p) = pM$

- Note

  $T(p) = \lambda T(p)$, $\lambda \neq 0$

Example:
Intersection of Parallels



Perspective Mapping

$$T(p) = (x_1, x_2, gx_1 + hx_2)$$

- Affine Points $\leftrightarrow$ Ideal Points
- Vanishing Points

  $p \notin \pi \rightarrow p' \in \pi$

  $||$ lines intersect in p'
- Principal Vanishing Points

Anatomy of a Matrix

$$M = \begin{pmatrix} A & P \\ T & S \end{pmatrix}$$

A - Affine Block

T - Translation Block

P - Perspective Block

S - Scaling Block

Affine Map

- leave π invariant

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation

- leave π invariant

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e & f & 1 \end{pmatrix}$$

Perspective

- don't leave π invariant

$$\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix}$$

Transformations Again

$p = (x_1, x_2, 1)$

- Projective Transformation

  $p' = Mp$

- Homogeneous Division

  $p' = \frac{1}{x'_3}(x'_1, x'_2, x'_3)$

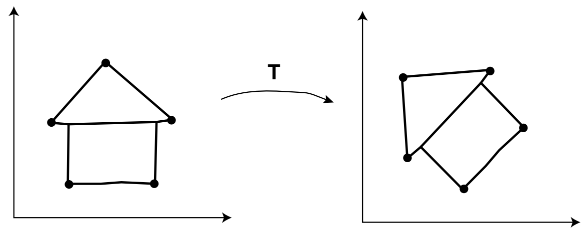
$p = (x''_1, x''_2, 1)$

Advantages of Projective Geometry

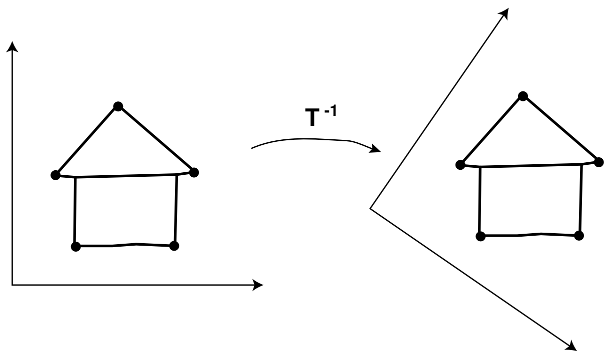
- Incorporate
  - Affine Mappings
  - Translations
  - Perspective
- Matrix Operations

Dual Interpretation

- Coordinate Transform
  - Change Points



- Mapping between Spaces
  - Change Coordinate Systems



Composition of Transformations

$$p' = T_n(\cdots T_2(T_1(p)))$$

$$p' = pM_1M_2\cdots M_n$$

$$p' = pM$$