Implicit Objects for Computer Graphics

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Motivation

- State of the Art Overview

  Consolidation of the Area

  Intense Reseach Activities

  - Workshop of Implicit Surfaces (1995, 1996, 1998)
  - SIGGRAPH Courses and Papers
  - Recent Books

- Review of VISGRAF Research

  Main Research Area

  People: jonas, lvelho, lhf, ruben

  - Papers
  - Thesis
  - Book

References

Introduction to Implicit Surfaces

- Jules Bloomenthal (editor)
- Chandrajit Bajaj
- Marie-Paule Gascuel
- Brian Wyvill
- Geoff Wyvill

1997, Morgan-Kaufmann

Implicit Objects in Computer Graphics

- Jonas Gomes
- Luiz Velho

1992, IMPA / 1998, Springer-Verlag

Outline of the Course

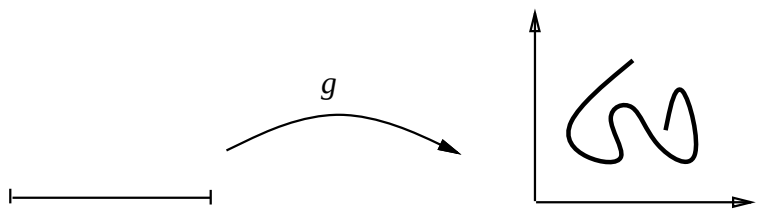
- Introduction and Overview
  - Basic Concepts (Book)
- Symbolic Methods
  - Algebraic Varieties and Symbolic Perturbations (J. Canny)
- Numeric Methods
  - PL Approximation and Interval Methods (L. Velho, L. Figueiredo, J. Gomes)
- Representation and Modeling
  - Multiresolution Models and Conversion (L. Velho, J. Gomes)
- Visualization and Animation
  - Particle Systems: Texturing and Morphing (Velho, Figueiredo, Gomes, Zonenschein, Wyvil)
- Survey of Recent Research
  - Groups and Papers

Basic Concepts

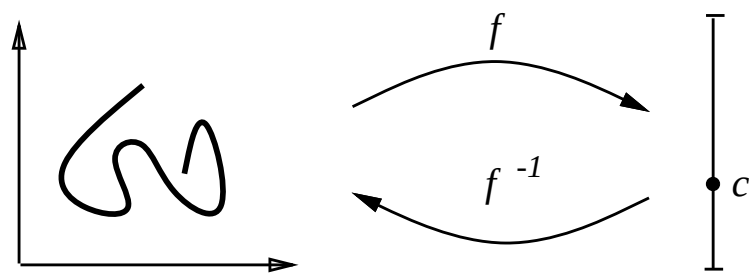
- Shape Description
  - Parametric and Implict Forms
  - Implicit Manifolds
- Defining Implicit Objects
  - Properties
  - Mathematical Characterization
  - Computational Implementation
- Operations with Implicit Objects
  - Modification
  - Combination

Shape Description

- Parametric Description
  $g : \mathbf{R}^m \rightarrow \mathbf{R}^n$



- Implicit Description
  $f : \mathbf{R}^n \rightarrow \mathbf{R}^{(n-m)}$

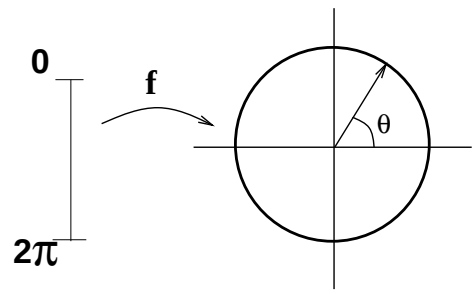


  where
  - m: dimension of the object
  - n: dimension of ambient space

* *Direct* versus *Indirect* Representation

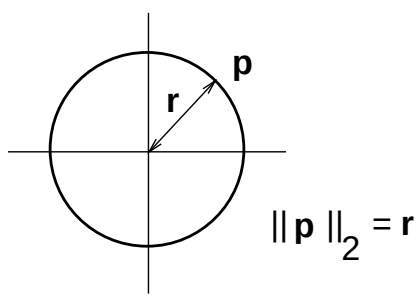
Examples

- Parametric



$$S^1 = \{(x, y) : x = \cos\theta, y = \sin\theta, \theta \in [0, 2\pi]\}$$

- Implicit



$$S^1 = \{(x, y) : x^2 + y^2 = 1, x, y \in \mathbf{R}^2\}$$

Comparison: Parametric Implicit

- Parametric
  - More Popular
  - Intrinsic Description

- Implicit
  - Gaining Acceptance
  - Extrinsic Description
  - Unifies Geometric and Volumetric Modeling
  - Suitable for Constraint Definitions

- Basic Operations
  - Enumeration of Points
  - Classification of Points

Pros and Cons

	Parametric	Implicit
Enumeration	+	-
Classification	-	+

* *Complementary Descriptions*

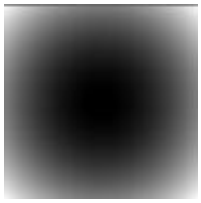
Implicit Model

- Implicit Function
  - Space Decomposition
  - Metrics

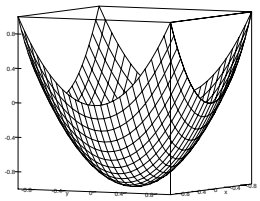
- Gradient of F
  - Regularity
  - Orientability

Intuitive View

- Implicit Function *(Density)*

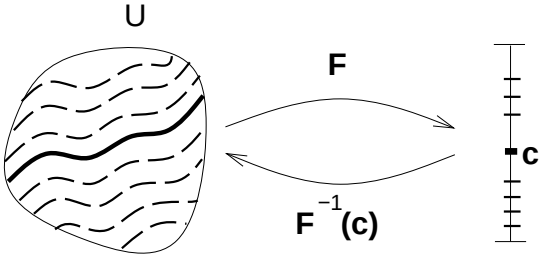


volume data

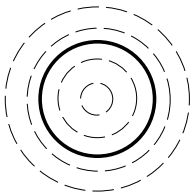


function graph

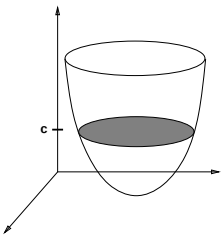
- Implicit Surface



inverse image of c



level curves

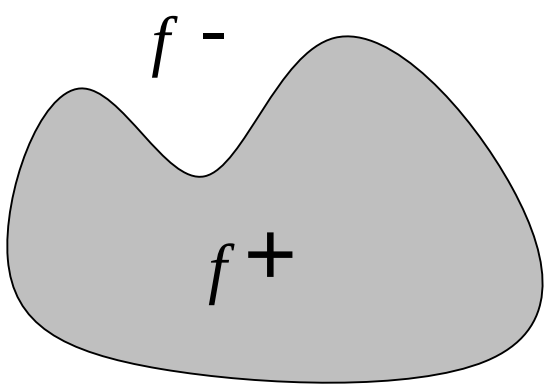


intersection

Space Decomposition

- f is a Point-Membership Classification Function

$$f(p) - c \begin{cases} > 0 & p \in \text{exterior points} \\ = 0 & p \in \text{boundary points} \\ < 0 & p \in \text{interior points} \end{cases}$$



- Characteristic Function of a Shape $\mathcal{S}$

  $\chi : \mathbf{R}^n \to [0, 1]$

$$\chi(p) \begin{cases} 1 & p \in \mathcal{S} \\ 0 & p \notin \mathcal{S} \end{cases}$$

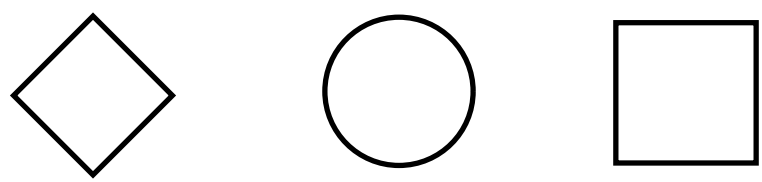
F and Metrics

$F(p)$ indicates *algebraic (signed) distance:*
$p \in \mathbf{R}^n$ to $F^{-1}(c)$
induced by a pseudo-metric $d : \mathbf{R}^n \to R$

- m-norms

$$d_m(p,q) = \left[\sum_i ||p_i, q_i||^m \right]^{1/m}$$

- ex: circle



d0 *d1* *d oo*

- Properties of F
  - Invariance
    $G(x) = \alpha F(x)$ *for* $F^{-1}(0)$
    $G(x) = F(x)^\alpha$ *for* $F^{-1}(1)$
  - Redundancy
    F *given;* $G^{-1}(c) \equiv F^{-1}(c)$
  - Contours
    $H(x) = F(x) - \delta$

Regularity

- Regular Value c of $F : R^n \to R$

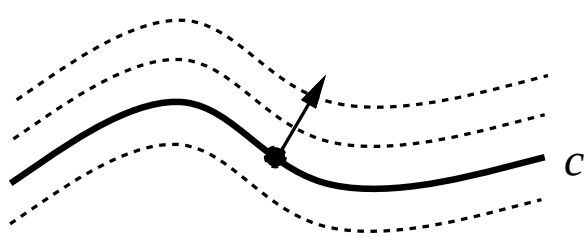
$$F' \text{ is surjective, } \forall x \in F^{-1}(c)$$

  i.e.

$$(\frac{\partial F}{x_1}, \ldots, \frac{\partial F}{x_n}) \neq 0, \forall x \in F^{-1}(c)$$

- $F(p) = c$ defines a *Differentiable Implicit Manifold*
  - F is a differentiable function
  - c is a *regular value* of F

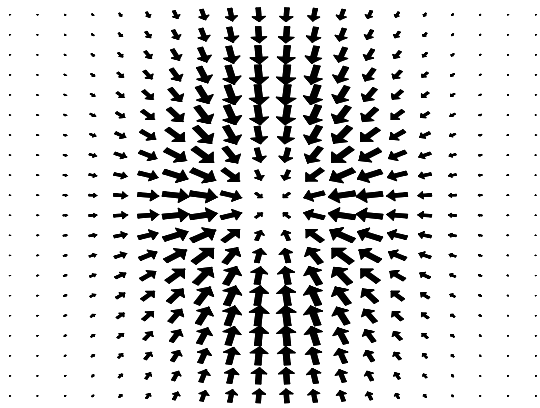
- Transversality



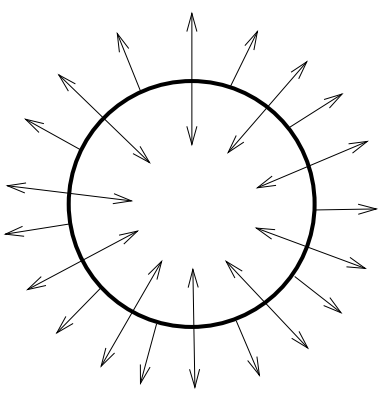
* *Sards Theorem:*
Can always satisfy regularity (perturbation)

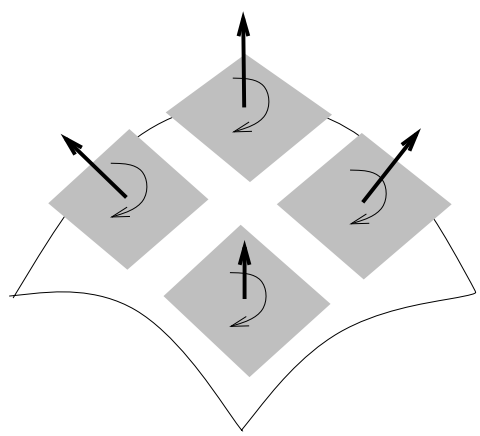
Orientability

- $\nabla F = (\frac{\partial F}{x_1}, \ldots, \frac{\partial F}{x_n})$



- Normal Vector Fields: $N_1(p) = -N_2(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}$





* $F^{-1}(c)$ is an *Orientable* Manifold

Implicit Representation

- Definition
  - Surfaces and Solids
  - Canonical Forms

- Characterization
  - Mathematical
  - Computational

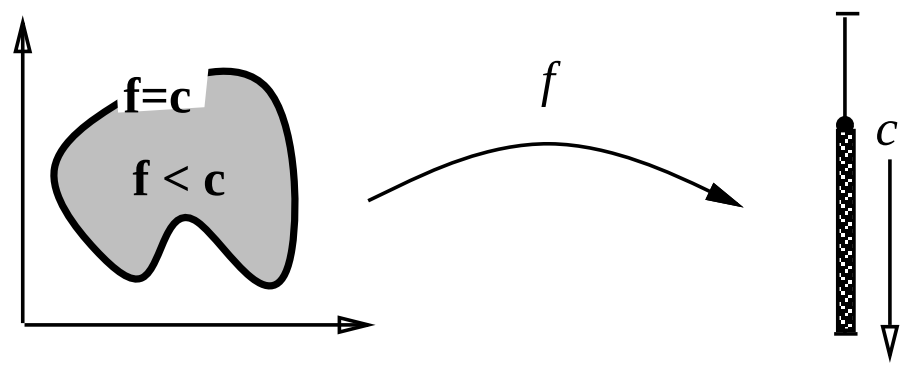
Definition

- Implicit Object

$$\mathcal{O} = (f, A)$$

where $f : \mathbf{R}^n \to \mathbf{R}$, and $A = [c, -\infty]$

- Solid Shape: $f(x) \leq c$



  - Boundary
    $f(x) = c$, codimension 1 manifold
  - Interior
    $f(x) < c$, submanifold of $\mathbf{R}^n$

Canonical Forms

$$\mathcal{O}_0 = (F_0, c_0)$$
$$F_0 : \mathbf{R}^n \to (-\infty, \infty)$$
$$c_0 = 0$$

$$\mathcal{O}_1 = (F_1, c_1)$$
$$F_1 : \mathbf{R}^n \to (0, \infty)$$
$$c_1 = 1$$

- Converting from $\mathcal{O}_0$ to $\mathcal{O}_1$

$$F_0(x) = \exp(F_1(x))$$

- Converting from $\mathcal{O}_1$ to $\mathcal{O}_0$

$$F_1(x) = \log(F_0(x))$$

* *Important in Applications*
$F_0^{-1}(0) \longleftrightarrow F_1^{-1}(1)$

Mathematical Characterization

- Implicit Function

$$F : \mathbf{R}^3 \to \mathbf{R}$$

- Gradient Vector Field

$$\nabla F(p) = (\frac{\partial F}{x_1}, \ldots, \frac{\partial F}{x_n})$$

- Normal Field

$$N = \frac{\nabla F}{|\nabla F|}$$

- Gauss Map (Hessian)

$$N'_x$$

  - Normal Curvature at x, in the direction v $N'_x(v) \cdot v$
  - Principal Directions (Eigenvectors of N')
  - Principal Curvatures (Eigenvalues of N')
  - Mean Curvature (Trace of N')
  - Gaussian Curvature (Determinant of N')

Computational Characterization

- Object-Oriented Approach
  - Parameters (Local State)
  - Procedures (System Interface)

- Public Functions
  - Bounding Box
    (domain of F)
  - Range and Value
    (R and c)
  - Point Classification
    (implicit function F)
  - Normal
    (gradient of F)
  - Curvature
    (hessian)

- General vs. Particular
  - Object Classes
  - Instancing

Operations with Implicit Objects

- Modification
  Unary Ops

$$F : \mathbf{R}^3 \rightarrow \mathbf{R}$$

  - Range Operations
    - *Metric (Density) Transformations*
  - Domain Operations
    - *Spatial Transformation*

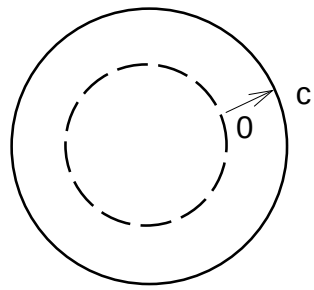
- Combination
  N-ary Ops

$$C(F_1(x), \ldots, F_k(x))$$

  - Blend Operations
    - *Smooth Joining*
  - Point-Set Operations
    - *CSG Combination*

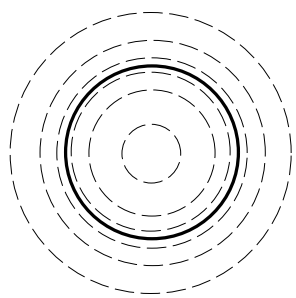
Range Operations

- Dilatations and Erosions



offset$(c, F(x)) = F(x) - c$

- Density Change

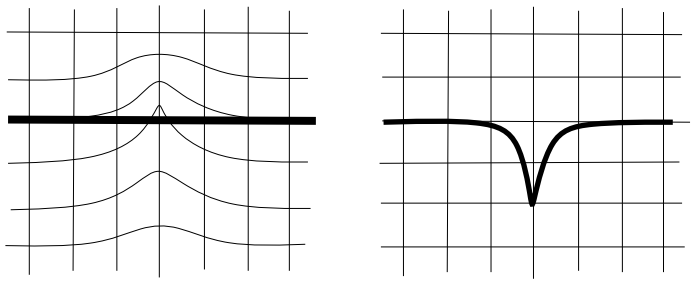


density$_m(d, F(x)) = d * F(x)$
density$_p(d, F(x)) = F(x)^d$

- Complement
  $\overline{F(x)} = -F(x)$

Domain Operations

$T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$



obs: inverse transformation T^{-1}

- Affine Mappings

  $T(x) = L(x) + v$

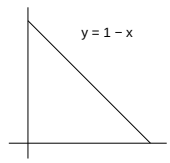
  - Rigid Motions
- Deformations

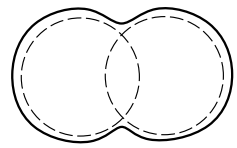
  W injective, W' non-singular

  - Simple Warps
    - fold, pull
  - Extrinsic Deformations
    - taper, bend, twist
  - Free Form Deformations
- Physically Based Deformations

Blend Operations

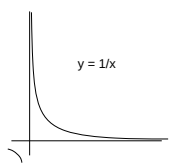
- Linear Blend

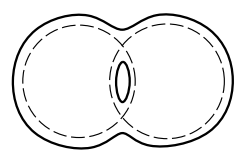




$$B_L(F_1(x), \ldots, F_k(x)) = \sum_{i=1}^{k} F_i(x) - 1$$

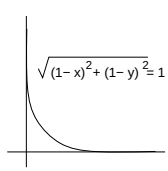
- Hyperbolic Blend

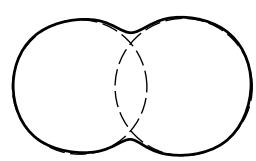




$$B_H(F_1(x), \ldots, F_k(x)) = \prod_{i=1}^{k} F_i(x) - 1$$

- Super-Elliptic Blend





$$B_S(F_1(x), \ldots, F_k(x)) = 1 - \left(\sum \max(0, 1 - F_i(x))^{\beta}\right)^{\frac{1}{\beta}}$$

Point-Set Operations

- Boolean Operations
  - Union
    $F_1 \cup F_2 = \min(F_1, F_2)$
  - Intersection
    $F_1 \cap F_2 = \max(F_1, F_2)$
  - Difference
    $F_1 \setminus F_2 = F_1 \cap \overline{F_2} = \max(F_1, -F_2)$

- Differentiable Booleans
  - Union
    $\lim_{p \to \infty} (F_1^p + F_2^p)^{\frac{1}{p}} = \min(F_1, F_2)$
  - Intersection
    $\lim_{p \to \infty} (F_1^{-p} + F_2^{-p})^{-\frac{1}{p}} = \max(F_1, F_2)$

Computational Methods

- Basic Procedures
  - Sampling (root finding)

    geometry
    - One Solution

      Intersection with a Line
    - Many Solutions

      Dynamic Particles
  - Structuring

    topology
    - Subdivide Space
      - Domain of F
    - Subdivide Object
      - Solid Region
      - Boundary

- Approximations
  - PL Approximation
  - Higher Order Approximations

Modeling

- Model Description
  - Object Representation
    - Primitives
    - Composite
  - Groups
    - Geometric Constraints
    - Other Purposes
  - Auxiliary Structures
    - Space Subdivision Enumeration
    - Polygonization

- Modeling Techniques
  - Constructive
  - Free-Form
  - Physically Based

- Conversion
  - Parametric - Implicit
  - Exact vs. Approximate

Primitive Implicit Objects

- Analytic
  - Plane
  - Quadrics / Tori
  - Superquadrics

- Skeletons
  - Points
  - Curves
  - Surfaces

- Volumetric
  - Scattered Data
  - Voxel Array

- Procedural
  - Hypertextures
  - Fractals

* Multiresolution

Visualization

- Points
  - Particles
- Curves
  - Discontinuity Lines
  - Contour Lines
- Surfaces
  - Scan Line
  - Ray Tracing
  - Polygonal
- Volumes
  - Projection
  - Ray Casting
  - * Slice Rendering

Texture Mapping

- Mapping Diagram $\quad g \circ t : V \subset \mathbf{R}^n \to \mathbf{R}^k$

$$\begin{tikzcd}
V \arrow[d, "g"'] \arrow[dr, "g \circ t"] & \\
U \arrow[r, "t"] & \mathbf{R}^k
\end{tikzcd}$$

- Spaces

  $V \subset \mathbf{R}^n$: object space

  $U \subset \mathbf{R}^m$: texture space

  $\mathbf{R}^k$: attribute space

- Functions

  $t : U \subset \mathbf{R}^m \to \mathbf{R}^k$: texture function

  $g : V \subset \mathbf{R}^n \to U$: mapping function

Types of Texture Mapping

- Texture Attribute
  - Color
  - Surface Properties
  - Displacement
  - Density

- Texture Space
  - 2D texture ($m = 2$)
  - 3D texture ($m = 3$)

- Surface Representation
  - Parametric $p : U \to V$
    (2D texture) $g = p^{-1}$
  - Implicit $f : V \to \mathbf{R}$
    (3D texture) $g = I$

Animation

- Skeletons
  - Particle Systems
  - Articulated Objects

- Metamorphosis
  - Interpolation
    $tF_1 + (1-t)F_2, \qquad t \in [0,1]$
  - Correspondence

- Dynamics

- Collision Detection

Conclusion

Advantages

- Representation of Surfaces & Volumes
- Description of Solids, Liquids, Gases
- Flexible Formulation for CAD/CAM
- Resolution and Levels of Detail
- Higher Dimensional Problems

- Topics
  - Computation
  - Modeling
  - Visualization
  - Animation

- Current Research