Multiresolution Meshes

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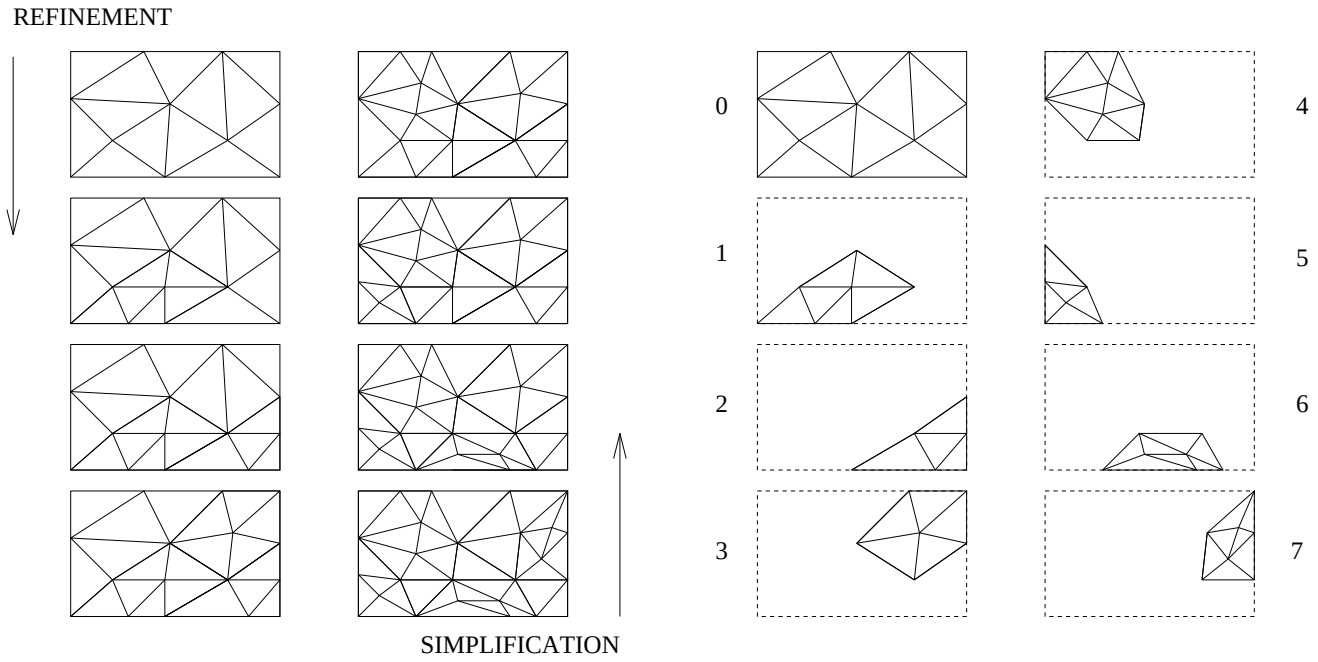
Outline

- Definitions
  - Fragments
  - Compatible Triangulations
  - Multi-Triangulation
- Properties
  - Expressiveness
  - Resolution Control
  - Algorithm Complexity
- Operations
  - Mesh Extraction
  - Structure Navigation
  - Spatial Search
- Representation
  - Data Structures
  - Pyramids, Trees, and Sequences

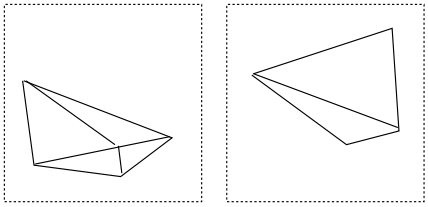
Intuition

- Initial Mesh + Modifications



- Depedency Relationships (Partial Order)

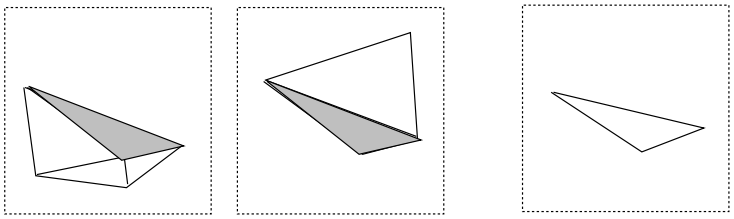
Operations with Fragments



T_i T_j

- Interference

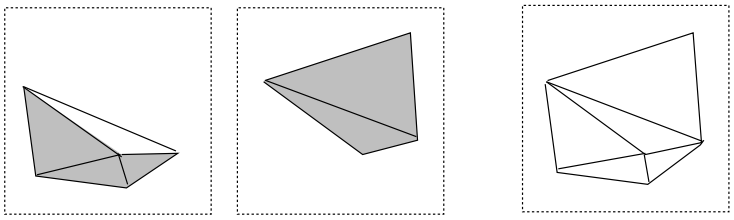
$$T_i \otimes T_j = \{t \in T_i | \exists t' \in T_j, i(t) \cap t' \neq \emptyset\}$$



T_i T_j $T_i \otimes T_j$

- Combination (Pasting)

$$T_i \oplus T_j = T_i \setminus (T_i \otimes T_j) \cup T_j$$



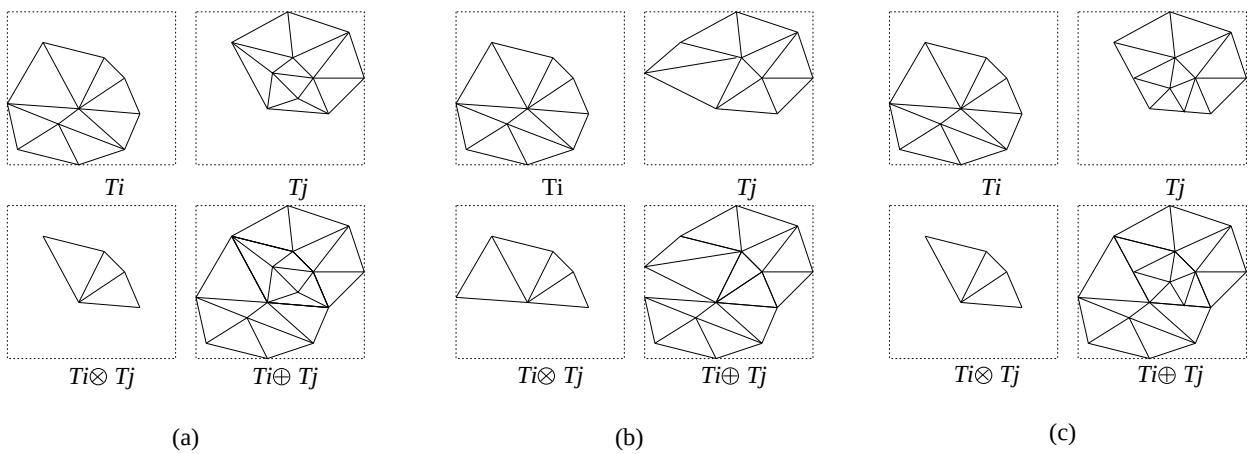
$T_i \setminus (T_i \otimes T_j)$ T_j $T_i \oplus T_j$

* *non-commutative, non-associative*

Compatible Triangulations

- Compatibility of T_j over T_i
  1. $T_i \oplus T_j$ is a triangulation
  2. $\Delta(T_i \oplus T_j) = \Delta(T_i) \cup \Delta(T_j)$

- Example: (a) OK, (b) fail 2, (c) fail 1.



- Compatible Sequence $T_0, T_1, \ldots T_k$
  - Pasting: $\oplus_{i=0}^{k} T_i = T_0 \oplus T_1 \oplus \cdots \oplus T_k$
    $\forall j = 1, \ldots, k,$
    T_j is compatible over $\oplus_{i=0}^{j-1} T_i$

Multi-Triangulation

- MT on Ω is a poset $(\mathcal{T}, \leq)$, where

  $\mathcal{T} = \{T_0, \ldots T_h\}$, is a set of triangulations, and

  $\leq$ is a partial order satisfying:

  1. $\Delta(T_0) \equiv \Omega$

     $\Delta(T_i) \subseteq \Omega, i = 1, \ldots h$
  2. $\forall i, j = 0, \ldots h,\ i \neq j$

     (a) $T_i \prec T_j \Rightarrow T_i \otimes T_j \neq \emptyset$

     (b) $T_i \otimes T_j \neq \emptyset \Rightarrow T_i < T_j$ or $T_j < T_i$
  3. $T_0, \ldots \mathcal{T}_h$

     · *Consistent* Total Order of $\mathcal{T}$

     · is a *Compatible Sequence*

- Associate t-set: $T_{\mathcal{T}} = \cup_{i=0}^{h} T_i$

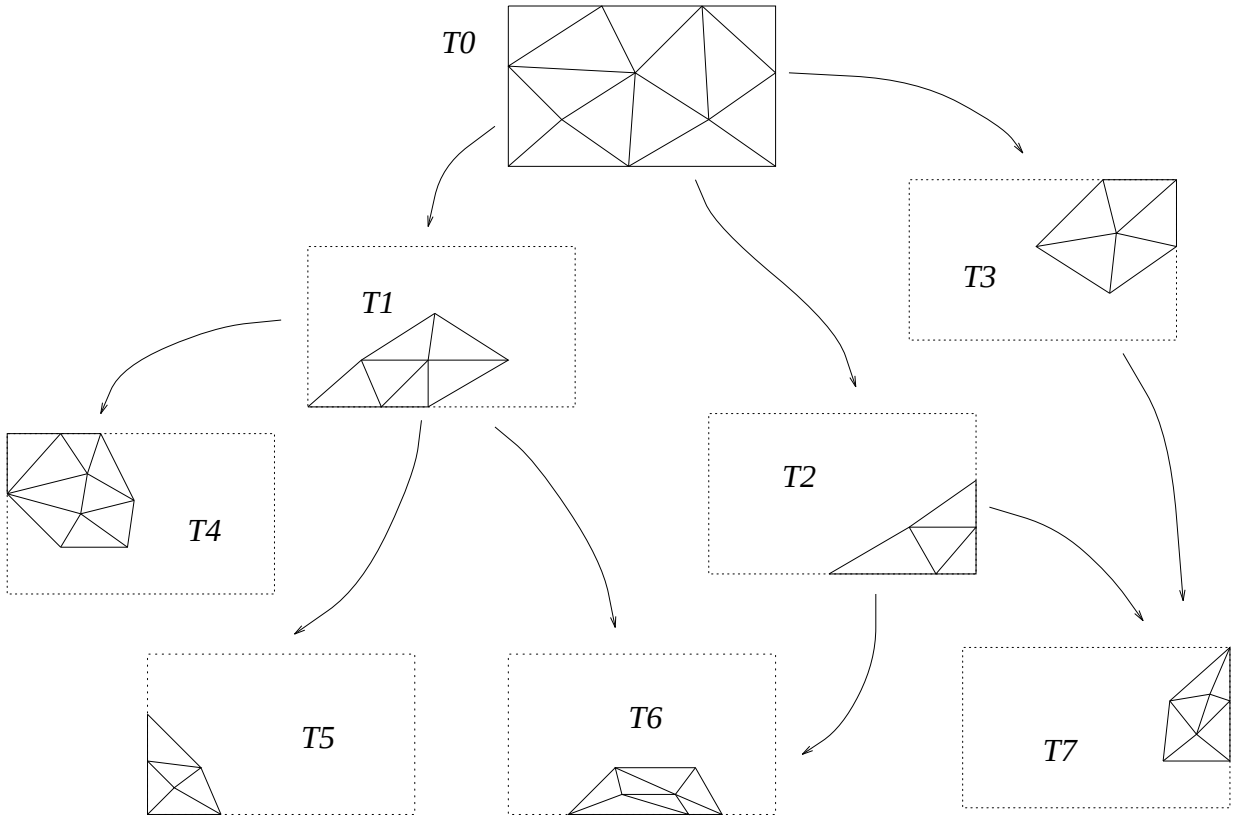
- Default Order

  Total Order of $\mathcal{T}' \subset \mathcal{T}$,

  consistent with $T_0, \ldots T_h$

Representation of MT

- Direct Acyclic Graph (DAG)
  - Nodes: Fragments T_i
  - Arcs: Precedence $\prec$



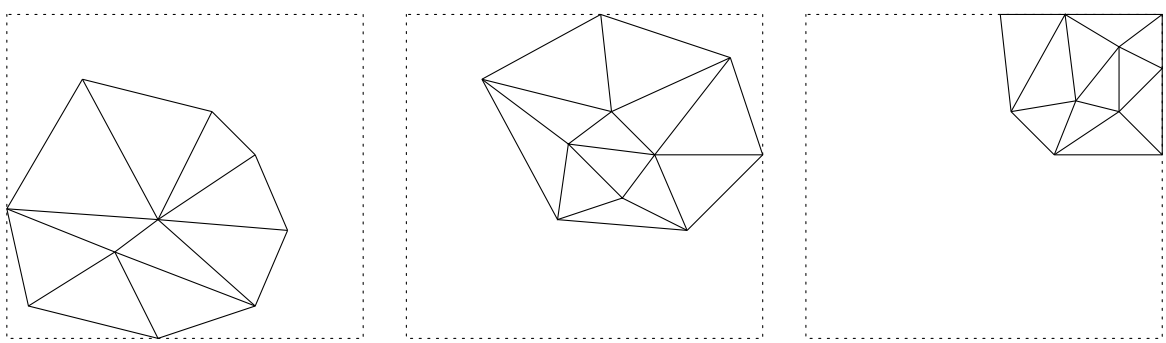
Building Triangulations

- Succesive Pasting

$$\oplus\mathcal{T}' = \oplus T_{i=0}^{k} = T_0 \oplus \cdots \oplus T_k$$

- Order Relation $\Rightarrow$ Fragment Dependency
  - if $T_i < T_j$
    T_i depends of T_J

- Precedence does not imply Interference

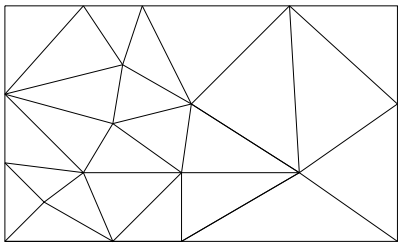


T1 *T2* *T3*

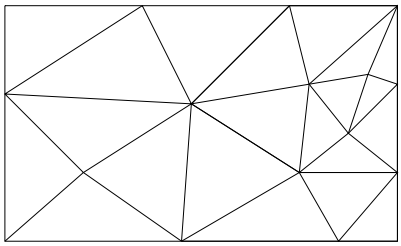
$T_1 \prec T_2 \prec T_3$
hence $T_1 < T_3$
but $T_1 \otimes T_3 = \emptyset$

Examples

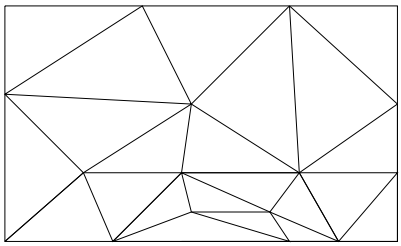
- 4 out of the 25 coverings of MT



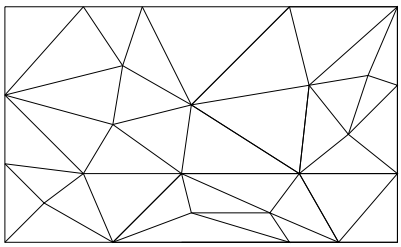
$T0 \oplus T1 \oplus T4 \oplus T5$



$T0 \oplus T2 \oplus T3 \oplus T7$



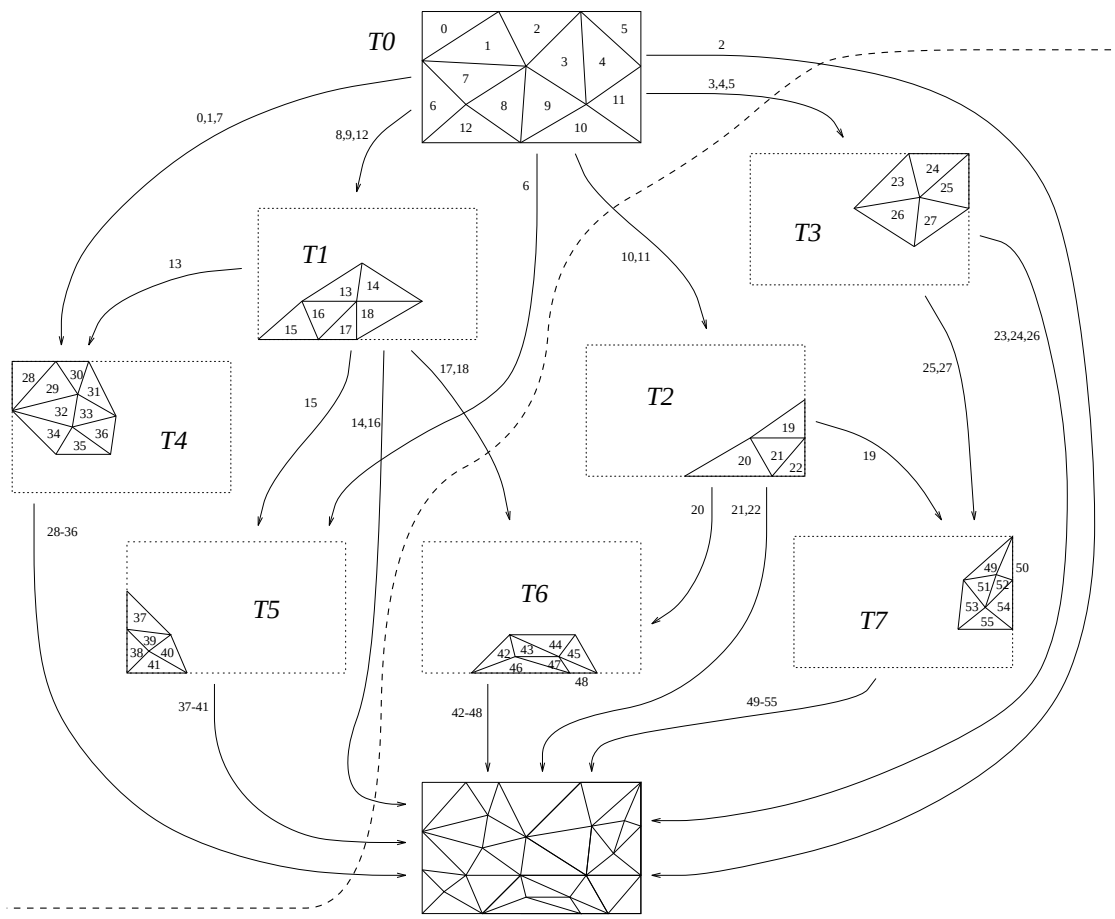
$T0 \oplus T1 \oplus T2 \oplus T6$



$T0 \oplus T1 \oplus T2 \oplus T3 \oplus$
$T4 \oplus T5 \oplus T6 \oplus T7$

Mesh Extraction

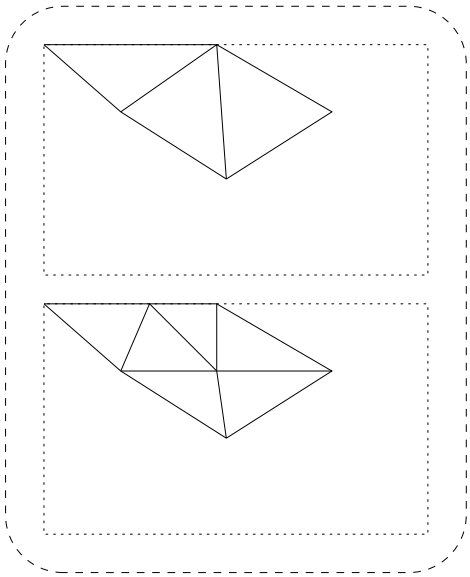
- Cut in the DAG
  $M = \oplus \mathcal{T}'$, $\mathcal{T}'$ is a lower set of $\mathcal{T}$



- Uniqueness and Corecteness
  - Default Order of $\mathcal{T}'$ is a compatible sequence
  - The t-set of $\mathcal{T}'$ is independent of order

Elements

- Fragment
  - t-set: T_i
  - Floor(T_i): $\oplus\mathcal{T}_{\overline{T_i}} \otimes T_I$



- Lattice
  - Least: T_0
    (empty Floor)
  - Top: $\oplus\mathcal{T}$
    (empty t-set)

Properties of MT

- Expressive Power

  Number of Meshes Generated by an MT

  - Canonical Form
  - Non-Redundancy

- Monotonicity

  Control over Resolution by Cuts

  - Increasing (Decreasing) MT
  - Reverse

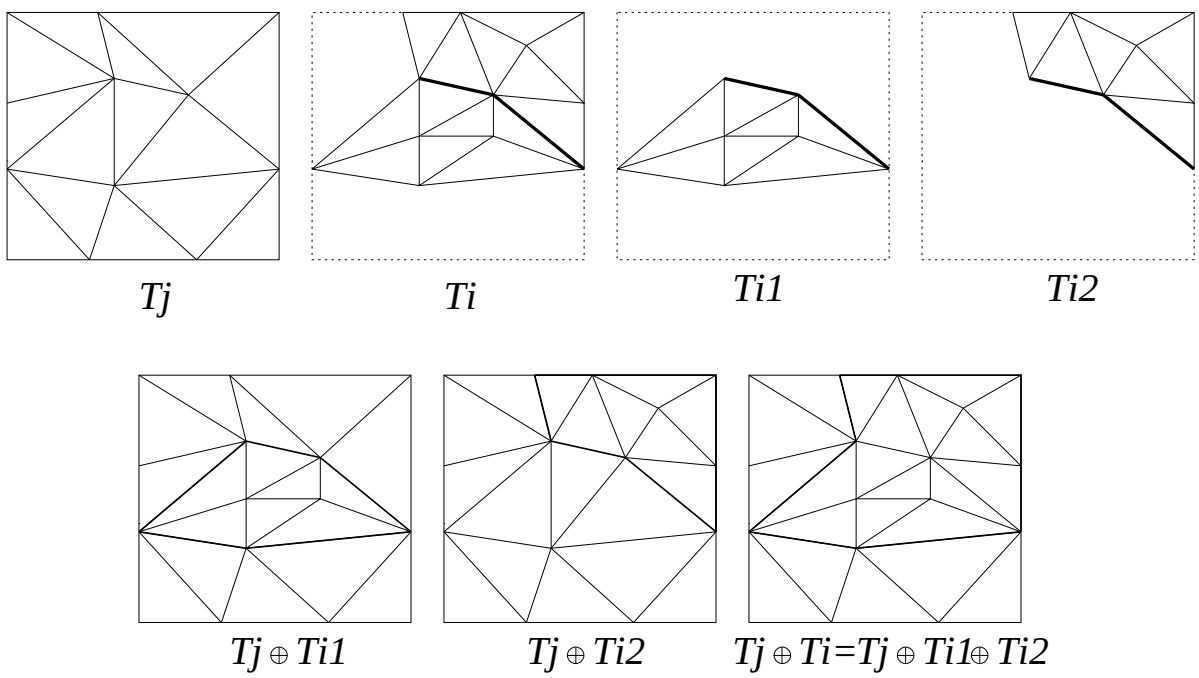
- Structure and Size of MT

  Space and Time Complexity

  - Growth
  - Width and Height

Canonical Form

- Minimal Compatibility
  - T_I compatible over T_j
  - $\nexists T_i' \subset T_i$ compatible over T_j



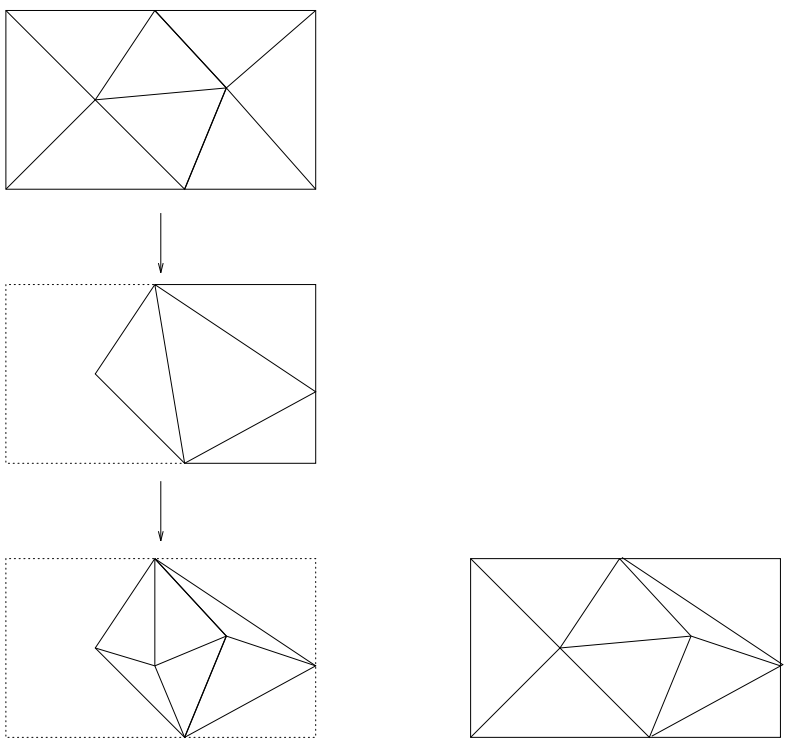
- Canonical Form

$$\mathcal{T} = \{T_i \in \mathcal{T} | T_i \text{ is minimally compatible}\}$$

obs: Any $\mathcal{T}$ can be put in Canonical Form

Expressiveness

- Non-Redundancy
  1. No Duplicates Triangles $t \in T_i \Rightarrow t \notin T_j$
  2. $\forall i, j = 0, h$ if e is a *common edge* of T_i, T_j with $T_j < T_i$, then $e \in$ Floor(T_i)



(a) (b)

If $\mathcal{T}$ is *Non-Redundant* and in *Canonical Form*, then, for any triangulation T generated from $T_{\mathcal{T}}$ $\exists$ a lower set $\mathcal{T}' \subseteq \mathcal{T}$ s.t. $T = \oplus \mathcal{T}'$

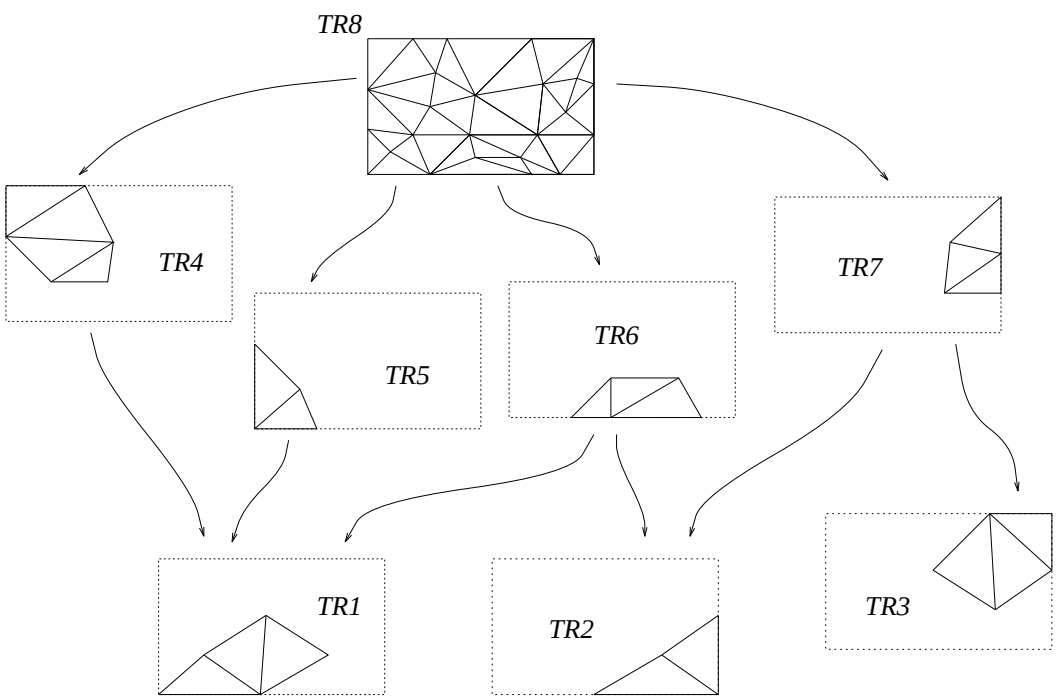
Monotonicity

- Increasing (Decreasing)

  $\mathcal{T}$ is increasing [decreasing],iff for every $\mathcal{T}'$, $\mathcal{T}''$

  $(\mathcal{T}' \subset \mathcal{T}'') \Rightarrow |\oplus \mathcal{T}'| < [>] \quad |\oplus \mathcal{T}''|$

- Reverse of $(\mathcal{T}, \leq)$ is $(\mathcal{T}^R, \leq^R)$
  1. $\mathcal{T}^R = \{T_1^R \ldots T_{h+1}^R\}, \quad T_i^R =$ Floor(T_i)
  2. $T_i \prec T_j \Rightarrow T_j^R \prec^R T_i^R$



- $\mathcal{T}$ is monotone iff $\mathcal{T}^R$ is monotone

Structure

- Linear Growth

  for each lower set $\mathcal{T}' \subseteq \mathcal{T}$,

  $|\mathcal{T}'|/|\oplus\mathcal{T}'| < C$

  $\mathcal{T}$ has linear growth if, $\forall T_i \in \mathcal{T}$, (1) or (2) is true

  1. $|\text{Floor}(T_i)| < K$
  2. $|T_i| < k|\text{floor}(T_i)|$

- Bounded Width

  # of arcs outgoing from T_i is bounded by C

- Logarithmic Height
  - Max Path from Least to Top $\approx$ log(# arcs of $\mathcal{T}$)

- Conservative
  - floor(T) has no internal vertices

** *Optimal Time Complexity of Algorithms visiting $\mathcal{T}$*

Variable Resolution Operations

Resolution Control Function

$$c : t \in \oplus\mathcal{T} \rightarrow [0, 1]$$

- Adapted Representation
  (*Mesh Extraction*)
  - Fixed Condition
  - Incremental Condition

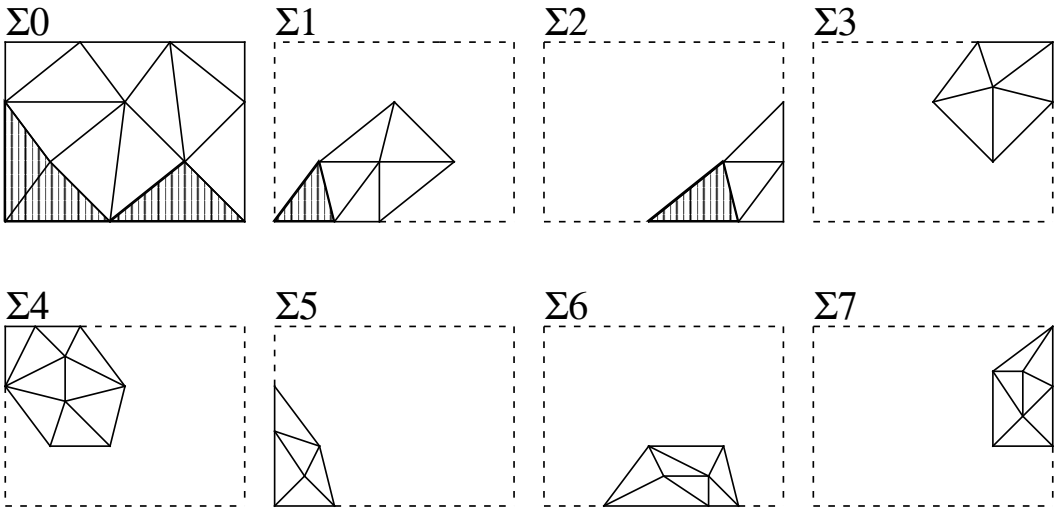
- Spatial Search
  (*Interference Relations*)
  - Coincidence
  - Inclusion
  - Intersection

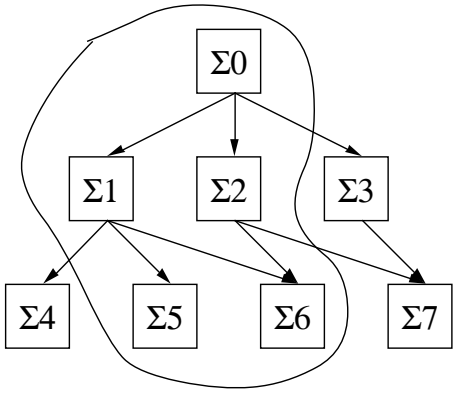
- Structure Navigation
  (*Topological Relations*)
  - Adjacency
  - Boundary
  - Co-Boundary

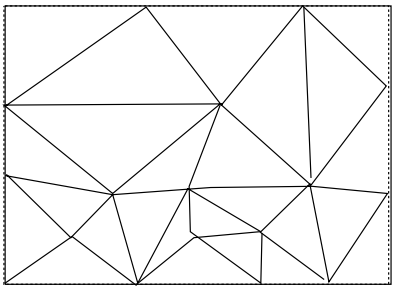
Mesh Extraction

Less refined mesh satisfying c:

1. $c(\oplus \mathcal{T}_c) = 1$
2. $\forall \mathcal{T}' \subset \mathcal{T}_c, c(\oplus \mathcal{T}') = 0$





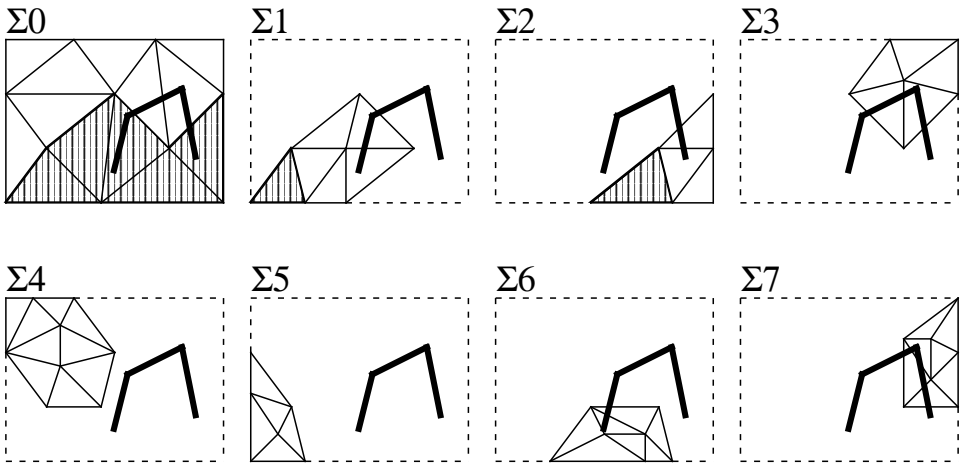


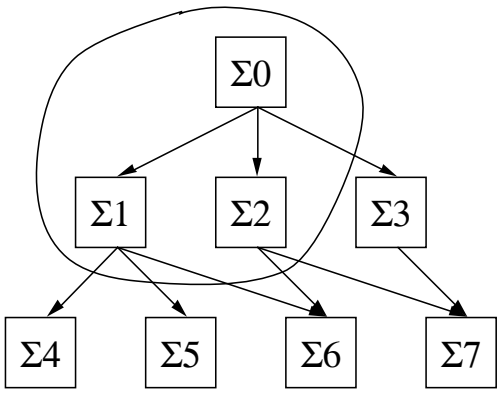
Σ0 ⊕Σ1 ⊕Σ2 ⊕Σ5 ⊕ Σ6

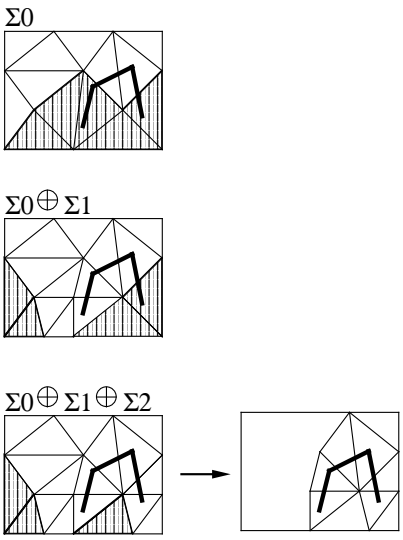
Spatial Search

- Query Entity: $q \subset \mathbf{R}^n$

- Interference Relation (intersection)

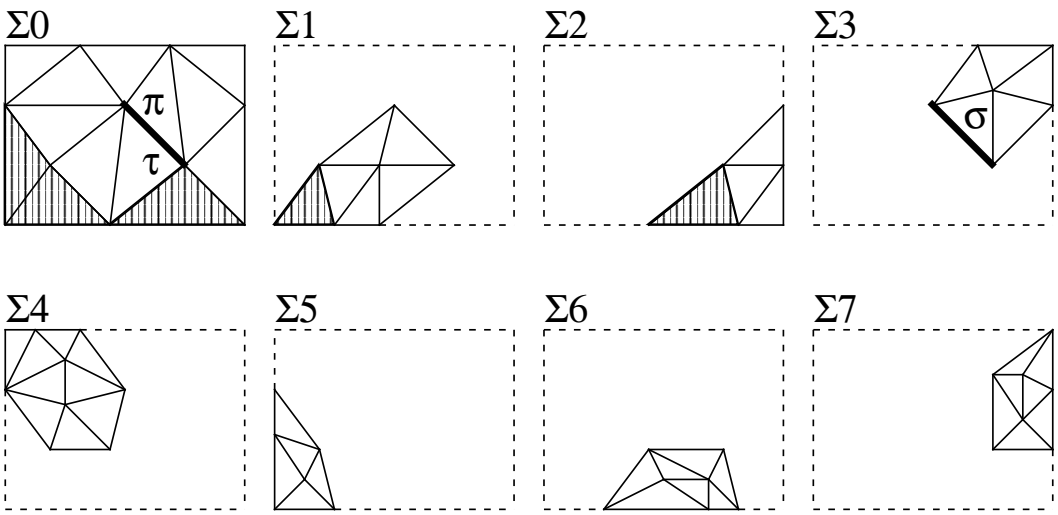






Structure Navigation

- Reference Simplex: ρ
- Topological Relation: R



Comparison with other Hierarchical Structures

- Pyramids
  - Global LOD
  - $\mathcal{E}$ Constant

    ex: Quaternary Triangulations

- Restricted Trees
  - Adapted LOD
  - $\mathcal{E}$ Constant, Fixed Transition

    ex: Restricted Quadtrees

- Progressive Sequences
  - Local LOD
  - $\mathcal{E}$ Variable, Fixed Order

    ex: Progrssive Meshes, Delaunay Sequences

OBS: Conversion to MT Framework