Chapter 3

Manifolds

3.1 Charts and Manifolds

In Chapter 1 we defined the notion of a manifold embedded in some ambient space, $\mathbb{R}^N$. In order to maximize the range of applications of the theory of manifolds it is necessary to generalize the concept of a manifold to spaces that are not a priori embedded in some $\mathbb{R}^N$. The basic idea is still that, whatever a manifold is, it is a topological space that can be covered by a collection of open subsets, U_α, where each U_α is isomorphic to some "standard model", *e.g.*, some open subset of Euclidean space, $\mathbb{R}^n$. Of course, manifolds would be very dull without functions defined on them and between them. This is a general fact learned from experience: *Geometry arises not just from spaces but from spaces and interesting classes of functions between them.* In particular, we still would like to "do calculus" on our manifold and have good notions of curves, tangent vectors, differential forms, etc. The small drawback with the more general approach is that the definition of a tangent vector is more abstract. We can still define the notion of a curve on a manifold, but such a curve does not live in any given $\mathbb{R}^n$, so it it not possible to define tangent vectors in a simple-minded way using derivatives. Instead, we have to resort to the notion of chart. This is not such a strange idea. For example, a geography atlas gives a set of maps of various portions of the earth and this provides a very good description of what the earth is, without actually imagining the earth embedded in 3-space.

The material of this chapter borrows from many sources, including Warner [145], Berger and Gostiaux [17], O'Neill [117], Do Carmo [50, 49], Gallot, Hulin and Lafontaine [60], Lang [95], Schwartz [133], Hirsch [76], Sharpe [137], Guillemin and Pollack [69], Lafontaine [92], Dubrovin, Fomenko and Novikov [52] and Boothby [18]. A nice (not very technical) exposition is given in Morita [112] (Chapter 1). The recent book by Tu [143] is also highly recommended for its clarity. Among the many texts on manifolds and differential geometry, the book by Choquet-Bruhat, DeWitt-Morette and Dillard-Bleick [37] stands apart because it is one of the clearest and most comprehensive (many proofs are omitted, but this can be an advantage!) Being written for (theoretical) physicists, it contains more examples and applications than most other sources.

Given $\mathbb{R}^n$, recall that the projection functions, $pr_i\colon \mathbb{R}^n \to \mathbb{R}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n.$$

For technical reasons (in particular, to ensure the existence of partitions of unity, see Section 3.6) and to avoid "esoteric" manifolds that do not arise in practice, from now on, all topological spaces under consideration will be assumed to be Hausdorff and second-countable (which means that the topology has a countable basis).

Definition 3.1 Given a topological space, M, a *chart* (or *local coordinate map*) is a pair, (U, φ), where U is an open subset of M and $\varphi\colon U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geq 1$). For any $p \in M$, a chart, (U, φ), is a *chart at* p iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of* p w.r.t. the chart. The inverse, (Ω, φ^{-1}), of a chart is called a *local parametrization*. Given any two charts, (U_i, φ_i) and (U_j, φ_j), if $U_i \cap U_j \neq \emptyset$, we have the *transition maps*, $\varphi_i^j\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_j^i\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, defined by

$$\varphi_i^j = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_j^i = \varphi_i \circ \varphi_j^{-1}.$$

Clearly, $\varphi_j^i = (\varphi_i^j)^{-1}$. Observe that the transition maps φ_i^j (resp. φ_j^i) are maps between *open subsets of* $\mathbb{R}^n$. This is good news! Indeed, the whole arsenal of calculus is available for functions on $\mathbb{R}^n$, and we will be able to promote many of these results to manifolds by imposing suitable conditions on transition functions.

Definition 3.2 Given a topological space, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a C^k *n-atlas* (or *n-atlas of class* C^k), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$ for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism. When $k = \infty$, the φ_i^j are smooth diffeomorphisms.

We must ensure that we have enough charts in order to carry out our program of generalizing calculus on $\mathbb{R}^n$ to manifolds. For this, we must be able to add new charts whenever necessary, provided that they are consistent with the previous charts in an existing atlas. Technically, given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the altas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$).

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* iff every chart of one is compatible with the other atlas. This is equivalent to saying that the union of the two atlases is still an atlas. It is immediately verified that compatibility induces an equivalence relation on C^k n-atlases on M. In fact, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$. Finally, we have our generalized notion of a manifold.

Definition 3.3 Given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k-manifold of dimension n* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M. Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on* $\mathbb{R}^n$. When $k = \infty$, we say that M is a *smooth manifold*.

Remark: It might have been better to use the terminology *abstract manifold* rather than manifold, to emphasize the fact that the space M is not a priori a subspace of $\mathbb{R}^N$, for some suitable N.

We can allow $k = 0$ in the above definitions. In this case, condition (3) in Definition 3.2 is void, since a C^0-diffeomorphism is just a homeomorphism, but φ_i^j is always a homeomorphism. In this case, M is called a *topological manifold of dimension n*. We do not require a manifold to be connected but we require all the components to have the same dimension, n. Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same. This is quite easy to show if $k \geq 1$ but for $k = 0$, this requires a deep theorem of Brouwer. (Brouwer's *Invariance of Domain Theorem* states that if $U \subseteq \mathbb{R}^n$ is an open set and if $f\colon U \to \mathbb{R}^n$ is a continuous and injective map, then $f(U)$ is open in $\mathbb{R}^n$. Using Brouwer's Theorem, we can show the following fact: If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are two open subsets and if $f\colon U \to V$ is a homeomorphism between U and V, then $m = n$. If $m > n$, then consider the injection, $i\colon \mathbb{R}^n \to \mathbb{R}^m$, where $i(x) = (x, 0_{m-n})$. Clearly, i is injective and continuous, so $f \circ i\colon U \to i(V)$ is injective and continuous and Brouwer's Theorem implies that $i(V)$ is open in $\mathbb{R}^m$, which is a contradiction, as $i(V) = V \times \{0_{m-n}\}$ is not open in $\mathbb{R}^m$. If $m < n$, consider the homeomorphism $f^{-1}\colon V \to U$.)

What happens if $n = 0$? In this case, every one-point subset of M is open, so every subset of M is open, i.e., M is any (countable if we assume M to be second-countable) set with the discrete topology!

Observe that since $\mathbb{R}^n$ is locally compact and locally connected, so is every manifold (check this!).

Remark: In some cases, M does not come with a topology in an obvious (or natural) way and a slight variation of Definition 3.2 is more convenient in such a situation:

Definition 3.4 Given a set, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) Each U_i is a subset of M and $\varphi_i \colon U_i \to \varphi_i(U_i)$ is a bijection onto an open subset, $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the sets $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are open in $\mathbb{R}^n$ and the transition maps φ_i^j and φ_j^i are C^k-diffeomorphisms.

Then, the notion of a chart being compatible with an atlas and of two atlases being compatible is just as before and we get a new definition of a manifold, analogous to Definition 3.3. But, this time, we give M the topology in which the open sets are arbitrary unions of domains of charts, U_i, more precisely, the U_i's of the maximal atlas defining the differentiable structure on M. It is not difficult to verify that the axioms of a topology are verified and M is indeed a topological space with this topology. It can also be shown that when M is equipped with the above topology, then the maps $\varphi_i \colon U_i \to \varphi_i(U_i)$ are homeomorphisms, so M is a manifold according to Definition 3.3. We also require that under this topology, M is Hausdorff and second-countable. A sufficient condition (in fact, also necessary!) for being second-countable is that some atlas be countable. A sufficient condition of M to be Hausdorff is that for all $p, q \in M$ with $p \neq q$, either $p, q \in U_i$ for some U_i or $p \in U_i$ and $q \in U_j$ for some disjoint U_i, U_j. Thus, we are back to the original notion of a manifold where it is assumed that M is already a topological space.

One can also define the topology on M in terms of any of the atlases, $\mathcal{A}$, defining M (not only the maximal one) by requiring $U \subseteq M$ to be open iff $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^n$, for every chart, (U_i, φ_i), in the altas $\mathcal{A}$. Then, one can prove that we obtain the same topology as the topology induced by the maximal atlas. For details, see Berger and Gostiaux [17], Chapter 2.

If the underlying topological space of a manifold is compact, then M has some finite atlas. Also, if $\mathcal{A}$ is some atlas for M and (U, φ) is a chart in $\mathcal{A}$, for any (nonempty) open subset, $V \subseteq U$, we get a chart, $(V, \varphi \restriction V)$, and it is obvious that this chart is compatible with $\mathcal{A}$. Thus, $(V, \varphi \restriction V)$ is also a chart for M. This observation shows that if U is any open subset of a C^k-manifold, M, then U is also a C^k-manifold whose charts are the restrictions of charts on M to U.

Example 1. The sphere S^n.

Using the stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $\sigma_N \colon S^n - \{N\} \to \mathbb{R}^n$ and $\sigma_S \colon S^n - \{S\} \to \mathbb{R}^n$, where $N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ (the north pole) and $S = (0, \cdots, 0, -1) \in \mathbb{R}^{n+1}$ (the south pole) be the maps called respectively *stereographic projection from the north pole* and *stereographic projection from the south pole* given by

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

Thus, if we let $U_N = S^n - \{N\}$ and $U_S = S^n - \{S\}$, we see that U_N and U_S are two open subsets covering S^n, both homeomorphic to $\mathbb{R}^n$. Furthermore, it is easily checked that on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2} (x_1, \ldots, x_n),$$

that is, the inversion of center $O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n - \{O\}$, so we conclude that (U_N, σ_N) and (U_S, σ_S) form a smooth atlas for S^n.

Example 2. The projective space $\mathbb{RP}^n$.

To define an atlas on $\mathbb{RP}^n$ it is convenient to view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

Given any $p = [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$, we call $(x_1, \ldots, x_{n+1})$ the *homogeneous coordinates* of p. It is customary to write $(x_1 \colon \cdots \colon x_{n+1})$ instead of $[x_1, \ldots, x_{n+1}]$. (Actually, in most books, the indexing starts with 0, i.e., homogeneous coordinates for $\mathbb{RP}^n$ are written as $(x_0 \colon \cdots \colon x_n)$.) For any i, with $1 \leq i \leq n+1$, let

$$U_i = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Observe that U_i is well defined, because if $(y_1 \colon \cdots \colon y_{n+1}) = (x_1 \colon \cdots \colon x_{n+1})$, then there is some $\lambda \neq 0$ so that $y_i = \lambda x_i$, for $i = 1, \ldots, n+1$. We can define a homeomorphism, φ_i, of U_i onto $\mathbb{R}^n$, as follows:

$$\varphi_i(x_1 \colon \cdots \colon x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

where the ith component is omitted. Again, it is clear that this map is well defined since it only involves ratios. We can also define the maps, ψ_i, from $\mathbb{R}^n$ to $U_i \subseteq \mathbb{RP}^n$, given by

$$\psi_i(x_1, \ldots, x_n) = (x_1 \colon \cdots \colon x_{i-1} \colon 1 \colon x_i \colon \cdots \colon x_n),$$

where the 1 goes in the ith slot, for $i = 1, \ldots, n+1$. One easily checks that φ_i and ψ_i are mutual inverses, so the φ_i are homeomorphisms. On the overlap, $U_i \cap U_j$, (where $i \neq j$), as $x_j \neq 0$, we have

$$(\varphi_j \circ \varphi_i^{-1})(x_1, \ldots, x_n) = \left(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

(We assumed that $i < j$; the case $j < i$ is similar.) This is clearly a smooth function from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$. As the U_i cover $\mathbb{RP}^n$, we conclude that the (U_i, φ_i) are $n+1$ charts making a smooth atlas for $\mathbb{RP}^n$. Intuitively, the space $\mathbb{RP}^n$ is obtained by glueing the open subsets U_i on their overlaps. Even for $n = 3$, this is not easy to visualize!

Example 3. The Grassmannian $G(k, n)$.

Recall that $G(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, also called k-planes. Every k-plane, W, is the linear span of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; furthermore, $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ both span W iff there is an invertible $k \times k$-matrix, $\Lambda = (\lambda_{ij})$, such that

$$v_j = \sum_{i=1}^{k} \lambda_{ij} u_i, \quad 1 \leq j \leq k.$$

Obviously, there is a bijection between the collection of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$ and the collection of $n \times k$ matrices of rank k. Furthermore, two $n \times k$ matrices A and B of rank k represent the same k-plane iff

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda.$$

(Note the analogy with projective spaces where two vectors u, v represent the same point iff $v = \lambda u$ for some invertible $\lambda \in \mathbb{R}$.) We can define the domain of charts (according to Definition 3.4) on $G(k, n)$ as follows: For every subset, $S = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, let U_S be the subset of $n \times k$ matrices, A, of rank k whose rows of index in $S = \{i_1, \ldots, i_k\}$ form an invertible $k \times k$ matrix denoted A_S. Observe that the $k \times k$ matrix consisting of the rows of the matrix AA_S^{-1} whose index belong to S is the identity matrix, I_k. Therefore, we can define a map, $\varphi_S \colon U_S \to \mathbb{R}^{(n-k)\times k}$, where $\varphi_S(A)$ is equal to the $(n-k) \times k$ matrix obtained by deleting the rows of index in S from AA_S^{-1}.

We need to check that this map is well defined, i.e., that it does not depend on the matrix, A, representing W. Let us do this in the case where $S = \{1, \ldots, k\}$, which is notationally simpler. The general case can be reduced to this one using a suitable permutation.

If $B = A\Lambda$, with Λ invertible, if we write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

as $B = A\Lambda$, we get $B_1 = A_1\Lambda$ and $B_2 = A_2\Lambda$, from which we deduce that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} B_1^{-1} = \begin{pmatrix} I_k \\ B_2 B_1^{-1} \end{pmatrix} = \begin{pmatrix} I_k \\ A_2 \Lambda \Lambda^{-1} A_1^{-1} \end{pmatrix} = \begin{pmatrix} I_k \\ A_2 A_1^{-1} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} A_1^{-1}.$$

Therefore, our map is indeed well-defined. It is clearly injective and we can define its inverse, ψ_S, as follows: Let π_S be the permutation of $\{1, \ldots, n\}$ swaping $\{1, \ldots, k\}$ and S and leaving every other element fixed (i.e., if $S = \{i_1, \ldots, i_k\}$, then $\pi_S(j) = i_j$ and $\pi_S(i_j) = j$, for $j = 1, \ldots, k$). If P_S is the permutation matrix associated with π_S, for any $(n-k) \times k$ matrix, M, let

$$\psi_S(M) = P_S \begin{pmatrix} I_k \\ M \end{pmatrix}.$$

The effect of ψ_S is to "insert into M" the rows of the identity matrix I_k as the rows of index from S. At this stage, we have charts that are bijections from subsets, U_S, of $G(k, n)$ to open subsets, namely, $\mathbb{R}^{(n-k)\times k}$. Then, the reader can check that the transition map $\varphi_T \circ \varphi_S^{-1}$ from $\varphi_S(U_S \cap U_U)$ to $\varphi_T(U_S \cap U_U)$ is given by

$$M \mapsto (C + DM)(A + BM)^{-1},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P_T P_S,$$

is the matrix of the permutation $\pi_T \circ \pi_S$ (this permutation "shuffles" S and T). This map is smooth, as it is given by determinants, and so, the charts (U_S, φ_S) form a smooth atlas for $G(k, n)$. Finally, one can check that the conditions of Definition 3.4 are satisfied, so the atlas just defined makes $G(k, n)$ into a topological space and a smooth manifold.

Remark: The reader should have no difficulty proving that the collection of k-planes represented by matrices in U_S is precisely the set of k-planes, W, supplementary to the $(n-k)$-plane spanned by the canonical basis vectors $e_{j_{k+1}}, \ldots, e_{j_n}$ (i.e., $\mathrm{span}(W \cup \{e_{j_{k+1}}, \ldots, e_{j_n}\}) = \mathbb{R}^n$, where $S = \{i_1, \ldots, i_k\}$ and $\{j_{k+1}, \ldots, j_n\} = \{1, \ldots, n\} - S$).

Example 4. Product Manifolds.

Let M_1 and M_2 be two C^k-manifolds of dimension n_1 and n_2, respectively. The topological space, $M_1 \times M_2$, with the product topology (the opens of $M_1 \times M_2$ are arbitrary unions of sets of the form $U \times V$, where U is open in M_1 and V is open in M_2) can be given the structure of a C^k-manifold of dimension $n_1 + n_2$ by defining charts as follows: For any two charts, (U_i, φ_i) on M_1 and (V_j, ψ_j) on M_2, we declare that $(U_i \times V_j, \varphi_i \times \psi_j)$ is a chart on $M_1 \times M_2$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^{n_1+n_2}$ is defined so that

$$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q)), \quad \text{for all } (p, q) \in U_i \times V_j.$$

We define C^k-maps between manifolds as follows:

Definition 3.5 Given any two C^k-manifolds, M and N, of dimension m and n respectively, a *C^k-map* is a continuous function, $h \colon M \to N$, satisfying the following property: For every $p \in M$, there is some chart, (U, φ), at p and some chart, (V, ψ), at $q = h(p)$, with $f(U) \subseteq V$ and

$$\psi \circ h \circ \varphi^{-1} \colon \varphi(U) \longrightarrow \psi(V)$$

a C^k-function.

It is easily shown that Definition 3.5 does not depend on the choice of charts. In particular, if $N = \mathbb{R}$, we obtain a C^k-*function on* M. One checks immediately that a function, $f\colon M \to \mathbb{R}$, is a C^k-map iff for every $p \in M$, there is some chart, (U, φ), at p so that

$$f \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \mathbb{R}$$

is a C^k-function. If U is an open subset of M, the set of C^k-functions on U is denoted by $\mathcal{C}^k(U)$. In particular, $\mathcal{C}^k(M)$ denotes the set of C^k-functions on the manifold, M. Observe that $\mathcal{C}^k(U)$ is a ring.

On the other hand, if M is an open interval of $\mathbb{R}$, say $M =]a, b[$, then $\gamma\colon]a, b[\to N$ is called a C^k-*curve* in N. One checks immediately that a function, $\gamma\colon]a, b[\to N$, is a C^k-map iff for every $q \in N$, there is some chart, (V, ψ), at q so that

$$\psi \circ \gamma\colon]a, b[\longrightarrow \psi(V)$$

is a C^k-function.

It is clear that the composition of C^k-maps is a C^k-map. A C^k-map, $h\colon M \to N$, between two manifolds is a C^k-*diffeomorphism* iff h has an inverse, $h^{-1}\colon N \to M$ (i.e., $h^{-1} \circ h = \mathrm{id}_M$ and $h \circ h^{-1} = \mathrm{id}_N$), and both h and h^{-1} are C^k-maps (in particular, h and h^{-1} are homeomorphisms). Next, we define tangent vectors.

3.2 Tangent Vectors, Tangent Spaces, Cotangent Spaces

Let M be a C^k manifold of dimension n, with $k \geq 1$. The most intuitive method to define tangent vectors is to use curves. Let $p \in M$ be any point on M and let $\gamma\colon]-\epsilon, \epsilon[\to M$ be a C^1-curve passing through p, that is, with $\gamma(0) = p$. Unfortunately, if M is not embedded in any $\mathbb{R}^N$, the derivative $\gamma'(0)$ does not make sense. However, for any chart, (U, φ), at p, the map $\varphi \circ \gamma$ is a C^1-curve in $\mathbb{R}^n$ and the tangent vector $v = (\varphi \circ \gamma)'(0)$ is well defined. The trouble is that different curves may yield the same v!

To remedy this problem, we define an equivalence relation on curves through p as follows:

Definition 3.6 Given a C^k manifold, M, of dimension n, for any $p \in M$, two C^1-curves, $\gamma_1\colon]-\epsilon_1, \epsilon_1[\to M$ and $\gamma_2\colon]-\epsilon_2, \epsilon_2[\to M$, through p (i.e., $\gamma_1(0) = \gamma_2(0) = p$) are *equivalent* iff there is some chart, (U, φ), at p so that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Now, the problem is that this definition seems to depend on the choice of the chart. Fortunately, this is not the case. For, if (V, ψ) is another chart at p, as p belongs both to U

and V, we have $U \cap V \neq 0$, so the transition function $\eta = \psi \circ \varphi^{-1}$ is C^k and, by the chain rule, we have

$$\begin{aligned}(\psi \circ \gamma_1)'(0) &= (\eta \circ \varphi \circ \gamma_1)'(0) \\ &= \eta'(\varphi(p))((\varphi \circ \gamma_1)'(0)) \\ &= \eta'(\varphi(p))((\varphi \circ \gamma_2)'(0)) \\ &= (\eta \circ \varphi \circ \gamma_2)'(0) \\ &= (\psi \circ \gamma_2)'(0).\end{aligned}$$

This leads us to the first definition of a tangent vector.

Definition 3.7 *(Tangent Vectors, Version 1)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to* M *at* p is any equivalence class of C^1-curves through p on M, modulo the equivalence relation defined in Definition 3.6. The set of all tangent vectors at p is denoted by $T_p(M)$ (or T_pM).

It is obvious that $T_p(M)$ is a vector space. If $u, v \in T_p(M)$ are defined by the curves γ_1 and γ_2, then $u + v$ is defined by the curve $\gamma_1 + \gamma_2$ (we may assume by reparametrization that γ_1 and γ_2 have the same domain.) Similarly, if $u \in T_p(M)$ is defined by a curve γ and $\lambda \in \mathbb{R}$, then λu is defined by the curve $\lambda\gamma$. The reader should check that these definitions do not depend on the choice of the curve in its equivalence class. We will show that $T_p(M)$ is a vector space of dimension n = dimension of M. One should observe that unless $M = \mathbb{R}^n$, in which case, for any $p, q \in \mathbb{R}^n$, the tangent space $T_q(M)$ is naturally isomorphic to the tangent space $T_p(M)$ by the translation $q - p$, for an arbitrary manifold, there is no relationship between $T_p(M)$ and $T_q(M)$ when $p \neq q$.

One of the defects of the above definition of a tangent vector is that it has no clear relation to the C^k-differential structure of M. In particular, the definition does not seem to have anything to do with the functions defined locally at p. There is another way to define tangent vectors that reveals this connection more clearly. Moreover, such a definition is more intrinsic, i.e., does not refer explicitly to charts. Our presentation of this second approach is heavily inspired by Schwartz [133] (Chapter 3, Section 9) but also by Warner [145].

As a first step, consider the following: Let (U, φ) be a chart at $p \in M$ (where M is a C^k-manifold of dimension n, with $k \geq 1$) and let $x_i = pr_i \circ \varphi$, the ith local coordinate $(1 \leq i \leq n)$. For any function, f, defined on $U \ni p$, set

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n.$$

(Here, $(\partial g/\partial X_i)|_y$ denotes the partial derivative of a function $g\colon \mathbb{R}^n \to \mathbb{R}$ with respect to the ith coordinate, evaluated at y.)

We would expect that the function that maps f to the above value is a linear map on the set of functions defined locally at p, but there is technical difficulty: The set of functions defined locally at p is **not** a vector space! To see this, observe that if f is defined on an open $U \ni p$ and g is defined on a different open $V \ni p$, then we do not know how to define $f + g$. The problem is that we need to identify functions that agree on a smaller open. This leads to the notion of *germs*.

Definition 3.8 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *locally defined function at* p is a pair, (U, f), where U is an open subset of M containing p and f is a function defined on U. Two locally defined functions, (U, f) and (V, g), at p are *equivalent* iff there is some open subset, $W \subseteq U \cap V$, containing p so that

$$f \restriction W = g \restriction W.$$

The equivalence class of a locally defined function at p, denoted $[f]$ or $\mathbf{f}$, is called a *germ at* p.

One should check that the relation of Definition 3.8 is indeed an equivalence relation. Of course, the value at p of all the functions, f, in any germ, $\mathbf{f}$, is $f(p)$. Thus, we set $\mathbf{f}(p) = f(p)$. One should also check that we can define addition of germs, multiplication of a germ by a scalar and multiplication of germs, in the obvious way: If $\mathbf{f}$ and $\mathbf{g}$ are two germs at p, and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} [f] + [g] &= [f + g] \\ \lambda[f] &= [\lambda f] \\ [f][g] &= [fg]. \end{aligned}$$

However, we have to check that these definitions make sense, that is, that they don't depend on the choice of representatives chosen in the equivalence classes $[f]$ and $[g]$. Let us give the details of this verification for the sum of two germs, $[f]$ and $[g]$. For any two locally defined functions, (f, U) and (g, V), at p, let $f + g$ be the locally defined function at p with domain $U \cap V$ and such that $(f + g)(x) = f(x) + g(x)$ for all $x \in U \cap V$. We need to check that for any locally defined functions (U_1, f_1), (U_2, f_2), (V_1, g_1), and (V_2, g_2), at p, if (U_1, f_1) and (U_2, f_2) are equivalent and if (V_1, g_1) and (V_2, g_2) are equivalent, then $(U_1 \cap V_1, f_1 + g_1)$ and $(U_2 \cap V_2, f_2 + g_2)$ are equivalent. However, as (U_1, f_1) and (U_2, f_2) are equivalent, there is some $W_1 \subseteq U_1 \cap U_2$ so that $f_1 \restriction W_1 = f_2 \restriction W_1$ and as (V_1, g_1) and (V_2, g_2) are equivalent, there is some $W_2 \subseteq V_1 \cap V_2$ so that $g_1 \restriction W_2 = g_2 \restriction W_2$. Then, observe that $(f_1 + g_1) \restriction (W_1 \cap W_2) = (f_2 + g_2) \restriction (W_1 \cap W_2)$, which means that $[f_1 + g_1] = [f_2 + g_2]$. Therefore, $[f + g]$ does not depend on the representatives chosen in the equivalence classes $[f]$ and $[g]$ and it makes sense to set

$$[f] + [g] = [f + g].$$

We can proceed in a similar fashion to define $\lambda[f]$ and $[f][g]$. Therefore, the germs at p form a ring. The ring of germs of C^k-functions at p is denoted $\mathcal{O}^{(k)}_{M,p}$. When $k = \infty$, we usually drop the superscript ∞.

Remark: Most readers will most likely be puzzled by the notation $\mathcal{O}^{(k)}_{M,p}$. In fact, it is standard in algebraic geometry, but it is not as commonly used in differential geometry. For any open subset, U, of a manifold, M, the ring, $\mathcal{C}^k(U)$, of C^k-functions on U is also denoted $\mathcal{O}^{(k)}_M(U)$ (certainly by people with an algebraic geometry bent!). Then, it turns out that the map $U \mapsto \mathcal{O}^{(k)}_M(U)$ is a *sheaf*, denoted $\mathcal{O}^{(k)}_M$, and the ring $\mathcal{O}^{(k)}_{M,p}$ is the *stalk* of the sheaf $\mathcal{O}^{(k)}_M$ at p. Such rings are called *local rings*. Roughly speaking, all the "local" information about M at p is contained in the local ring $\mathcal{O}^{(k)}_{M,p}$. (This is to be taken with a grain of salt. In the C^k-case where $k < \infty$, we also need the "stationary germs", as we will see shortly.)

Now that we have a rigorous way of dealing with functions locally defined at p, observe that the map

$$v_i \colon f \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f$$

yields the same value for all functions f in a germ $\mathbf{f}$ at p. Furthermore, the above map is linear on $\mathcal{O}^{(k)}_{M,p}$. More is true. Firstly for any two functions f, g locally defined at p, we have

$$\left(\frac{\partial}{\partial x_i}\right)_p (fg) = f(p)\left(\frac{\partial}{\partial x_i}\right)_p g + g(p)\left(\frac{\partial}{\partial x_i}\right)_p f.$$

Secondly, if $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, then

$$\left(\frac{\partial}{\partial x_i}\right)_p f = 0.$$

The first property says that v_i is a *derivation*. As to the second property, when $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, we say that f *is stationary at* p. It is easy to check (using the chain rule) that being stationary at p does not depend on the chart, (U, φ), at p or on the function chosen in a germ, $\mathbf{f}$. Therefore, the notion of a stationary germ makes sense: We say that $\mathbf{f}$ *is a stationary germ* iff $(f \circ \varphi^{-1})'(\varphi(p)) = 0$ for some chart, (U, φ), at p and some function, f, in the germ, $\mathbf{f}$. The C^k-stationary germs form a subring of $\mathcal{O}^{(k)}_{M,p}$ (but not an ideal!) denoted $\mathcal{S}^{(k)}_{M,p}$.

Remarkably, it turns out that the dual of the vector space, $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, is isomorphic to the tangent space, $T_p(M)$. First, we prove that the subspace of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ has $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ as a basis.

Proposition 3.1 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n functions, $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$, defined on $\mathcal{O}^{(k)}_{M,p}$ by*

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n$$

are linear forms that vanish on $\mathcal{S}^{(k)}_{M,p}$. Every linear form, L, on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$ can be expressed in a unique way as

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

where $\lambda_i \in \mathbb{R}$. Therefore, the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

form a basis of the vector space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

Proof. The first part of the proposition is trivial, by definition of $(f \circ \varphi^{-1})'(\varphi(p))$ and of $\left(\frac{\partial}{\partial x_i}\right)_p f$.

Next, assume that L is a linear form on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$. Consider the locally defined function at p given by

$$g(q) = f(q) - \sum_{i=1}^{n} (pr_i \circ \varphi)(q) \left(\frac{\partial}{\partial x_i}\right)_p f.$$

Observe that the germ of g is stationary at p, since

$$\begin{aligned} g(q) = (g \circ \varphi^{-1})(\varphi(q)) &= (f \circ \varphi^{-1})(\varphi(q)) - \sum_{i=1}^{n} (pr_i \circ \varphi)(q) \left(\frac{\partial}{\partial x_i}\right)_p f \\ &= (f \circ \varphi^{-1})(X_1(q) \ldots, X_n(q)) - \sum_{i=1}^{n} X_i(q) \left(\frac{\partial}{\partial x_i}\right)_p f, \end{aligned}$$

with $X_i(q) = (pr_i \circ \varphi)(q)$. It follows that

$$\left.\frac{\partial (g \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)} = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)} - \left(\frac{\partial}{\partial x_i}\right)_p f = 0.$$

But then, as L vanishes on stationary germs, we get

$$L(f) = \sum_{i=1}^{n} L(pr_i \circ \varphi) \left(\frac{\partial}{\partial x_i}\right)_p f,$$

as desired. We still have to prove linear independence. If

$$\sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p = 0,$$

then, if we apply this relation to the functions $x_i = pr_i \circ \varphi$, as

$$\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{ij},$$

we get $\lambda_i = 0$, for $i = 1, \ldots, n$. □

As the subspace of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ is isomorphic to the dual, $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$, of the space $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$, we see that the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

also form a basis of $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$.

To define our second version of tangent vectors, we need to define linear derivations.

Definition 3.9 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *linear derivation at* p is a linear form, v, on $\mathcal{O}_{M,p}^{(k)}$, such that

$$v(\mathbf{fg}) = f(p)v(\mathbf{g}) + g(p)v(\mathbf{f}),$$

for all germs $\mathbf{f}, \mathbf{g} \in \mathcal{O}_{M,p}^{(k)}$. The above is called the *Leibnitz property*.

Recall that we observed earlier that the $\left(\frac{\partial}{\partial x_i}\right)_p$ are linear derivations at p. Therefore, we have

Proposition 3.2 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ are exactly the linear derivations on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.*

Proof. By Proposition 3.1, the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

form a basis of the linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$. Since each $\left(\frac{\partial}{\partial x_i}\right)_p$ is a also a linear derivation at p, the result follows. □

Proposition 3.2 says that a linear form on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$ is a linear derivation but in general, when $k \neq \infty$, a linear derivation on $\mathcal{O}^{(k)}_{M,p}$ does *not* necessarily vanish on $\mathcal{S}^{(k)}_{M,p}$. However, we will see in Proposition 3.6 that this is true for $k = \infty$.

Here is now our second definition of a tangent vector.

Definition 3.10 *(Tangent Vectors, Version 2)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any linear derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$, the subspace of stationary germs.

Let us consider the simple case where $M = \mathbb{R}$. In this case, for every $x \in \mathbb{R}$, the tangent space, $T_x(\mathbb{R})$, is a one-dimensional vector space isomorphic to $\mathbb{R}$ and $\left(\frac{\partial}{\partial t}\right)_x = \frac{d}{dt}\Big|_x$ is a basis vector of $T_x(\mathbb{R})$. For every C^k-function, f, locally defined at x, we have

$$\left(\frac{\partial}{\partial t}\right)_x f = \frac{df}{dt}\bigg|_x = f'(x).$$

Thus, $\left(\frac{\partial}{\partial t}\right)_x$ is: compute the derivative of a function at x.

We now prove the equivalence of the two definitions of a tangent vector.

Proposition 3.3 *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For any $p \in M$, let u be any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma\colon]-\epsilon, +\epsilon[\to M$, through p (i.e., $p = \gamma(0)$). Then, the map L_u defined on $\mathcal{O}^{(k)}_{M,p}$ by*

$$L_u(\mathbf{f}) = (f \circ \gamma)'(0)$$

is a linear derivation that vanishes on $\mathcal{S}^{(k)}_{M,p}$. Furthermore, the map $u \mapsto L_u$ defined above is an isomorphism between $T_p(M)$ and $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$, the space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

Proof. Clearly, $L_u(\mathbf{f})$ does not depend on the representative, f, chosen in the germ, $\mathbf{f}$. If γ and σ are equivalent curves defining u, then $(\varphi \circ \sigma)'(0) = (\varphi \circ \gamma)'(0)$, so we get

$$(f \circ \sigma)'(0) = (f \circ \varphi^{-1})'(\varphi(p))((\varphi \circ \sigma)'(0)) = (f \circ \varphi^{-1})'(\varphi(p))((\varphi \circ \gamma)'(0)) = (f \circ \gamma)'(0),$$

which shows that $L_u(\mathbf{f})$ does not depend on the curve, γ, defining u. If $\mathbf{f}$ is a stationary germ, then pick any chart, (U, φ), at p and let $\psi = \varphi \circ \gamma$. We have

$$L_u(\mathbf{f}) = (f \circ \gamma)'(0) = ((f \circ \varphi^{-1}) \circ (\varphi \circ \gamma))'(0) = (f \circ \varphi^{-1})'(\varphi(p))(\psi'(0)) = 0,$$

since $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, as $\mathbf{f}$ is a stationary germ. The definition of L_u makes it clear that L_u is a linear derivation at p. If $u \neq v$ are two distinct tangent vectors, then there exist some curves γ and σ through p so that

$$(\varphi \circ \gamma)'(0) \neq (\varphi \circ \sigma)'(0).$$

Thus, there is some i, with $1 \leq i \leq n$, so that if we let $f = pr_i \circ \varphi$, then

$$(f \circ \gamma)'(0) \neq (f \circ \sigma)'(0),$$

and so, $L_u \neq L_v$. This proves that the map $u \mapsto L_u$ is injective.

For surjectivity, recall that every linear map, L, on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$ can be uniquely expressed as

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p .$$

Define the curve, γ, on M through p by

$$\gamma(t) = \varphi^{-1}(\varphi(p) + t(\lambda_1, \ldots, \lambda_n)),$$

for t in a small open interval containing 0. Then, we have

$$f(\gamma(t)) = (f \circ \varphi^{-1})(\varphi(p) + t(\lambda_1, \ldots, \lambda_n)),$$

and we get

$$(f \circ \gamma)'(0) = (f \circ \varphi^{-1})'(\varphi(p))(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i \left. \frac{\partial (f \circ \varphi^{-1})}{\partial X_i} \right|_{\varphi(p)} = L(\mathbf{f}).$$

This proves that $T_p(M)$ and $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$ are isomorphic. $\square$

In view of Proposition 3.3, we can identify $T_p(M)$ with $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$. As the space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ is finite dimensional, $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^{**}$ is canonically isomorphic to $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, so we can identify $T^*_p(M)$ with $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$. (Recall that if E is a finite dimensional space, the map $i_E \colon E \to E^{**}$ defined so that, for any $v \in E$,

$$v \mapsto \widetilde{v}, \quad \text{where} \quad \widetilde{v}(f) = f(v), \quad \text{for all } f \in E^*$$

is a linear isomorphism.) This also suggests the following definition:

Definition 3.11 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the *tangent space at* p, denoted $T_p(M)$ is the space of linear derivations on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$. Thus, $T_p(M)$ can be identified with $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$. The space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ is called the *cotangent space at* p; it is isomorphic to the dual, $T^*_p(M)$, of $T_p(M)$. (For simplicity of notation we also denote $T_p(M)$ by T_pM and $T^*_p(M)$ by T^*_pM.)

Even though this is just a restatement of Proposition 3.1, we state the following proposition because of its practical usefulness:

Proposition 3.4 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n tangent vectors,*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p,$$

form a basis of T_pM.

Observe that if $x_i = pr_i \circ \varphi$, as

$$\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{i,j},$$

the images of $x_1, \ldots, x_n$ in $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ form the dual basis of the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$. Given any C^k-function, f, on M, we denote the image of f in $T^*_p(M) = \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ by df_p. This is the *differential of f at p*. Using the isomorphism between $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ and $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^{**}$ described above, df_p corresponds to the linear map in $T^*_p(M)$ defined by $df_p(v) = v(\mathbf{f})$, for all $v \in T_p(M)$. With this notation, we see that $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T^*_p(M)$, and this basis is dual to the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$. For simplicity of notation, we often omit the subscript p unless confusion arises.

Remark: Strictly speaking, a tangent vector, $v \in T_p(M)$, is defined on the space of germs, $\mathcal{O}^{(k)}_{M,p}$, at p. However, it is often convenient to define v on C^k-functions, $f \in \mathcal{C}^k(U)$, where U is some open subset containing p. This is easy: Set

$$v(f) = v(\mathbf{f}).$$

Given any chart, (U, φ), at p, since v can be written in a unique way as

$$v = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

we get

$$v(f) = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p f.$$

This shows that $v(f)$ is the *directional derivative of f in the direction v*. The directional derivative, $v(f)$, is also denoted $v[f]$.

When M is a smooth manifold, things get a little simpler. Indeed, it turns out that in this case, every linear derivation vanishes on stationary germs. To prove this, we recall the following result from calculus (see Warner [145]):

Proposition 3.5 *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is a C^k-function ($k \geq 2$) on a convex open, U, about $p \in \mathbb{R}^n$, then for every $q \in U$, we have*

$$g(q) = g(p) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_p (q_i - p_i) + \sum_{i,j=1}^{n} (q_i - p_i)(q_j - p_j) \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{(1-t)p+tq} dt.$$

In particular, if $g \in C^\infty(U)$, then the integral as a function of q is C^∞.

Proposition 3.6 *Let M be any C^∞-manifold of dimension n. For any $p \in M$, any linear derivation on $\mathcal{O}^{(\infty)}_{M,p}$ vanishes on $\mathcal{S}^{(\infty)}_{M,p}$, the ring of stationary germs.*

Proof. Pick some chart, (U, φ), at p, where U is convex (for instance, an open ball) and let $\mathbf{f}$ be any stationary germ. If we apply Proposition 3.5 to $f \circ \varphi^{-1}$ and then compose with φ, we get

$$f = f(p) + \sum_{i=1}^{n} \left.\frac{\partial (f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)} (x_i - x_i(p)) + \sum_{i,j=1}^{n} (x_i - x_i(p))(x_j - x_j(p)) h,$$

near p, where h is C^∞. Since $\mathbf{f}$ is a stationary germ, this yields

$$f = f(p) + \sum_{i,j=1}^{n} (x_i - x_i(p))(x_j - x_j(p)) h.$$

If v is any linear derivation, we get

$$\begin{aligned} v(f) = v(f(p)) + \sum_{i,j=1}^{n} \Big[&(x_i - x_i(p))(p)(x_j - x_j(p))(p) v(h) \\ &+ (x_i - x_i(p))(p) v(x_j - x_j(p)) h(p) + v(x_i - x_i(p))(x_j - x_j(p))(p) h(p) \Big] = 0. \end{aligned}$$

Thus, v vanishes on stationary germs. $\square$

Proposition 3.6 shows that in the case of a smooth manifold, in Definition 3.10, we can omit the requirement that linear derivations vanish on stationary germs, since this is automatic. It is also possible to define $T_p(M)$ just in terms of $\mathcal{O}^{(\infty)}_{M,p}$. Let $\mathfrak{m}_{M,p} \subseteq \mathcal{O}^{(\infty)}_{M,p}$ be the ideal of germs that vanish at p. Then, we also have the ideal $\mathfrak{m}^2_{M,p}$, which consists of all finite sums of products of two elements in $\mathfrak{m}_{M,p}$, and it can be shown that $T^*_p(M)$ is isomorphic to $\mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}$ (see Warner [145], Lemma 1.16).

Actually, if we let $\mathfrak{m}^{(k)}_{M,p}$ denote the C^k germs that vanish at p and $\mathfrak{s}^{(k)}_{M,p}$ denote the stationary C^k-germs that vanish at p, it is easy to show that

$$\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p} \cong \mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p}.$$

(Given any $\mathbf{f} \in \mathcal{O}^{(k)}_{M,p}$, send it to $\mathbf{f} - \mathbf{f}(\mathbf{p}) \in \mathfrak{m}^{(k)}_{M,p}$.) Clearly, $(\mathfrak{m}^{(k)}_{M,p})^2$ consists of stationary germs (by the derivation property) and when $k = \infty$, Proposition 3.5 shows that every stationary germ that vanishes at p belongs to $\mathfrak{m}^2_{M,p}$. Therefore, when $k = \infty$, we have $\mathfrak{s}^{(\infty)}_{M,p} = \mathfrak{m}^2_{M,p}$ and so,

$$T^*_p(M) = \mathcal{O}^{(\infty)}_{M,p}/\mathcal{S}^{(\infty)}_{M,p} \cong \mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}.$$

Remark: The ideal $\mathfrak{m}^{(k)}_{M,p}$ is in fact the unique maximal ideal of $\mathcal{O}^{(k)}_{M,p}$. This is because if $\mathbf{f} \in \mathcal{O}^{(k)}_{M,p}$ does not vanish at p, then it is an invertible element of $\mathcal{O}^{(k)}_{M,p}$ and any ideal containing $\mathfrak{m}^{(k)}_{M,p}$ and $\mathbf{f}$ would be equal to $\mathcal{O}^{(k)}_{M,p}$, which it absurd. Thus, $\mathcal{O}^{(k)}_{M,p}$ is a local ring (in the sense of commutative algebra) called the *local ring of germs of C^k-functions at p*. These rings play a crucial role in algebraic geometry.

Yet one more way of defining tangent vectors will make it a little easier to define tangent bundles.

Definition 3.12 *(Tangent Vectors, Version 3)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, consider the triples, (U, φ, u), where (U, φ) is any chart at p and u is any vector in $\mathbb{R}^n$. Say that two such triples (U, φ, u) and (V, ψ, v) are *equivalent* iff

$$(\psi \circ \varphi^{-1})'_{\varphi(p)}(u) = v.$$

A *tangent vector* to M at p is an equivalence class of triples, $[(U, \varphi, u)]$, for the above equivalence relation.

The intuition behind Definition 3.12 is quite clear: The vector u is considered as a tangent vector to $\mathbb{R}^n$ at $\varphi(p)$. If (U, φ) is a chart on M at p, we can define a natural isomorphism, $\theta_{U,\varphi,p} \colon \mathbb{R}^n \to T_p(M)$, between $\mathbb{R}^n$ and $T_p(M)$, as follows: For any $u \in \mathbb{R}^n$,

$$\theta_{U,\varphi,p} \colon u \mapsto [(U, \varphi, u)].$$

One immediately checks that the above map is indeed linear and a bijection.

The equivalence of this definition with the definition in terms of curves (Definition 3.7) is easy to prove.

Proposition 3.7 *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For every $p \in M$, for every chart, (U, φ), at p, if x is any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma \colon]-\epsilon, +\epsilon[\to M$, through p (i.e., $p = \gamma(0)$), then the map*

$$x \mapsto [(U, \varphi, (\varphi \circ \gamma)'(0))]$$

is an isomorphism between $T_p(M)$-version 1 and $T_p(M)$-version 3.

Proof. If σ is another curve equivalent to γ, then $(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$, so the map is well-defined. It is clearly injective. As for surjectivity, define the curve, γ, on M through p by

$$\gamma(t) = \varphi^{-1}(\varphi(p) + tu).$$

Then, $(\varphi \circ \gamma)(t) = \varphi(p) + tu$ and

$$(\varphi \circ \gamma)'(0) = u.$$

□

After having explored thorougly the notion of tangent vector, we show how a C^k-map, $h\colon M \to N$, between C^k manifolds, induces a linear map, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, for every $p \in M$. We find it convenient to use Version 2 of the definition of a tangent vector. So, let $u \in T_p(M)$ be a linear derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$. We would like $dh_p(u)$ to be a linear derivation on $\mathcal{O}^{(k)}_{N,h(p)}$ that vanishes on $\mathcal{S}^{(k)}_{N,h(p)}$. So, for every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$, set

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{h}).$$

For any locally defined function, g, at $h(p)$ in the germ, $\mathbf{g}$ (at $h(p)$), it is clear that $g \circ h$ is locally defined at p and is C^k, so $\mathbf{g} \circ \mathbf{h}$ is indeed a C^k-germ at p. Moreover, if $\mathbf{g}$ is a stationary germ at $h(p)$, then for some chart, (V, ψ) on N at $q = h(p)$, we have $(g \circ \psi^{-1})'(\psi(q)) = 0$ and, for any chart, (U, φ), at p on M, we get

$$(g \circ h \circ \varphi^{-1})'(\varphi(p)) = (g \circ \psi^{-1})'(\psi(q))((\psi \circ h \circ \varphi^{-1})'(\varphi(p))) = 0,$$

which means that $\mathbf{g} \circ \mathbf{h}$ is stationary at p. Therefore, $dh_p(u) \in T_{h(p)}(M)$. It is also clear that dh_p is a linear map. We summarize all this in the following definition:

Definition 3.13 Given any two C^k-manifolds, M and N, of dimension m and n, respectively, for any C^k-map, $h\colon M \to N$, and for every $p \in M$, the *differential of h at p* or *tangent map*, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, is the linear map defined so that

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{h}),$$

for every $u \in T_p(M)$ and every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$. The linear map dh_p is also denoted T_ph (and sometimes, h'_p or D_ph).

The chain rule is easily generalized to manifolds.

Proposition 3.8 *Given any two C^k-maps $f\colon M \to N$ and $g\colon N \to P$ between smooth C^k-manifolds, for any $p \in M$, we have*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

In the special case where $N = \mathbb{R}$, a C^k-map between the manifolds M and $\mathbb{R}$ is just a C^k-function on M. It is interesting to see what df_p is explicitly. Since $N = \mathbb{R}$, germs (of functions on $\mathbb{R}$) at $t_0 = f(p)$ are just germs of C^k-functions, $g\colon \mathbb{R} \to \mathbb{R}$, locally defined at t_0. Then, for any $u \in T_p(M)$ and every germ $\mathbf{g}$ at t_0,

$$df_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{f}).$$

If we pick a chart, (U, φ), on M at p, we know that the $\left(\frac{\partial}{\partial x_i}\right)_p$ form a basis of $T_p(M)$, with $1 \leq i \leq n$. Therefore, it is enough to figure out what $df_p(u)(\mathbf{g})$ is when $u = \left(\frac{\partial}{\partial x_i}\right)_p$. In this case,

$$df_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left.\frac{\partial(g \circ f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}.$$

Using the chain rule, we find that

$$df_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left(\frac{\partial}{\partial x_i}\right)_p f \left.\frac{dg}{dt}\right|_{t_0}.$$

Therefore, we have

$$df_p(u) = u(\mathbf{f}) \left.\frac{d}{dt}\right|_{t_0}.$$

This shows that we can identify df_p with the linear form in $T_p^*(M)$ defined by

$$df_p(u) = u(\mathbf{f}), \qquad u \in T_pM,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$. This is consistent with our previous definition of df_p as the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ (as $T_p(M)$ is isomorphic to $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$).

Again, even though this is just a restatement of facts we already showed, we state the following proposition because of its practical usefulness:

Proposition 3.9 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n linear maps,*

$$(dx_1)_p, \ldots, (dx_n)_p,$$

form a basis of T_p^*M, where $(dx_i)_p$, the differential of x_i at p, is identified with the linear form in T_p^*M such that $(dx_i)_p(v) = v(\mathbf{x_i})$, for every $v \in T_pM$ (by identifying $T_\lambda\mathbb{R}$ with $\mathbb{R}$).

In preparation for the definition of the flow of a vector field (which will be needed to define the exponential map in Lie group theory), we need to define the tangent vector to a curve on a manifold. Given a C^k-curve, $\gamma\colon]a,b[\to M$, on a C^k-manifold, M, for any $t_0 \in]a,b[$, we would like to define the tangent vector to the curve γ at t_0 as a tangent vector to M at $p = \gamma(t_0)$. We do this as follows: Recall that $\frac{d}{dt}\big|_{t_0}$ is a basis vector of $T_{t_0}(\mathbb{R}) = \mathbb{R}$. So, define the *tangent vector to the curve* γ *at* t_0, denoted $\dot{\gamma}(t_0)$ (or $\gamma'(t_0)$, or $\frac{d\gamma}{dt}(t_0)$) by

$$\dot{\gamma}(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right).$$

Sometime, it is necessary to define curves (in a manifold) whose domain is not an open interval. A map, $\gamma\colon [a, b] \to M$, is a C^k*-curve in* M if it is the restriction of some C^k-curve, $\widetilde{\gamma}\colon]a-\epsilon, b+\epsilon[\to M$, for some (small) $\epsilon > 0$. Note that for such a curve (if $k \geq 1$) the tangent vector, $\dot{\gamma}(t)$, is defined for all $t \in [a, b]$. A continuous curve, $\gamma\colon [a, b] \to M$, is *piecewise* C^k iff there a sequence, $a_0 = a, a_1, \ldots, a_m = b$, so that the restriction, γ_i, of γ to each $[a_i, a_{i+1}]$ is a C^k-curve, for $i = 0, \ldots, m-1$. This implies that $\gamma_i'(a_{i+1})$ and $\gamma_{i+1}'(a_{i+1})$ are defined for $i = 0, \ldots, m-1$, but there may be a jump in the tangent vector to γ at a_i, that is, we may have $\gamma_i'(a_{i+1}) \neq \gamma_{i+1}'(a_{i+1})$.

3.3 Tangent and Cotangent Bundles, Vector Fields, Lie Derivative

Let M be a C^k-manifold (with $k \geq 2$). Roughly speaking, a vector field on M is the assignment, $p \mapsto X(p)$, of a tangent vector, $X(p) \in T_p(M)$, to a point $p \in M$. Generally, we would like such assignments to have some smoothness properties when p varies in M, for example, to be C^l, for some l related to k. Now, if the collection, $T(M)$, of all tangent spaces, $T_p(M)$, was a C^l-manifold, then it would be very easy to define what we mean by a C^l-vector field: We would simply require the map, $X\colon M \to T(M)$, to be C^l.

If M is a C^k-manifold of dimension n, then we can indeed make $T(M)$ into a C^{k-1}-manifold of dimension $2n$ and we now sketch this construction.

We find it most convenient to use Version 3 of the definition of tangent vectors, i.e., as equivalence classes of triples (U, φ, x), where (U, φ) is a chart and $x \in \mathbb{R}^n$. First, we let $T(M)$ be the disjoint union of the tangent spaces $T_p(M)$, for all $p \in M$. There is a *natural projection*,

$$\pi\colon T(M) \to M, \quad \text{where} \quad \pi(v) = p \quad \text{if} \quad v \in T_p(M).$$

We still have to give $T(M)$ a topology and to define a C^{k-1}-atlas. For every chart, (U, φ), of M (with U open in M) we define the function, $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$, by

$$\widetilde{\varphi}(v) = (\varphi \circ \pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v)),$$

where $v \in \pi^{-1}(U)$ and $\theta_{U,\varphi,p}$ is the isomorphism between $\mathbb{R}^n$ and $T_p(M)$ described just after Definition 3.12. It is obvious that $\widetilde{\varphi}$ is a bijection between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$, an open subset of $\mathbb{R}^{2n}$. We give $T(M)$ the weakest topology that makes all the $\widetilde{\varphi}$ continuous, i.e., we take the collection of subsets of the form $\widetilde{\varphi}^{-1}(W)$, where W is any open subset of $\varphi(U) \times \mathbb{R}^n$, as a basis of the topology of $T(M)$. One easily checks that $T(M)$ is Hausdorff and second-countable in this topology. If (U, φ) and (V, ψ) are overlapping charts, then the transition map,

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

is given by

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(z, x) = (\psi \circ \varphi^{-1}(z), (\psi \circ \varphi^{-1})'_z(x)), \qquad (z, x) \in \varphi(U \cap V) \times \mathbb{R}^n.$$

It is clear that $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a C^{k-1}-map. Therefore, $T(M)$ is indeed a C^{k-1}-manifold of dimension $2n$, called the *tangent bundle*.

Remark: Even if the manifold M is naturally embedded in $\mathbb{R}^N$ (for some $N \geq n = \dim(M)$), it is not at all obvious how to view the tangent bundle, $T(M)$, as embedded in $\mathbb{R}^{N'}$, for some suitable N'. Hence, we see that the definition of an abtract manifold is unavoidable.

A similar construction can be carried out for the cotangent bundle. In this case, we let $T^*(M)$ be the disjoint union of the cotangent spaces $T_p^*(M)$. We also have a natural projection, $\pi \colon T^*(M) \to M$, and we can define charts in several ways. One method used by Warner [145] goes as follows: For any chart, (U, φ), on M, we define the function, $\widetilde{\varphi} \colon \pi^{-1}(U) \to \mathbb{R}^{2n}$, by

$$\widetilde{\varphi}(\tau) = \left(\varphi \circ \pi(\tau), \tau\left(\left(\frac{\partial}{\partial x_1} \right)_{\pi(\tau)} \right), \ldots, \tau\left(\left(\frac{\partial}{\partial x_n} \right)_{\pi(\tau)} \right) \right),$$

where $\tau \in \pi^{-1}(U)$ and the $\left(\frac{\partial}{\partial x_i} \right)_p$ are the basis of $T_p(M)$ associated with the chart (U, φ). Again, one can make $T^*(M)$ into a C^{k-1}-manifold of dimension $2n$, called the *cotangent bundle*. We leave the details as an exercise to the reader (Or, look at Berger and Gostiaux [17]). Another method using Version 3 of the definition of tangent vectors is presented in Section 7.2. For simplicity of notation, we also use the notation TM for $T(M)$ (resp. T^*M for $T^*(M)$).

Observe that for every chart, (U, φ), on M, there is a bijection

$$\tau_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

given by

$$\tau_U(v) = (\pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v)).$$

Clearly, $pr_1 \circ \tau_U = \pi$, on $\pi^{-1}(U)$. Thus, locally, that is, over U, the bundle $T(M)$ looks like the product $U \times \mathbb{R}^n$. We say that $T(M)$ is *locally trivial* (over U) and we call τ_U a *trivializing map*. For any $p \in M$, the vector space $\pi^{-1}(p) = T_p(M)$ is called the *fibre above* p. Observe that the restriction of τ_U to $\pi^{-1}(p)$ is an isomorphism between $T_p(M)$ and $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$, for any $p \in M$. All these ingredients are part of being a *vector bundle* (but a little more is required of the maps τ_U). For more on bundles, see Chapter 7, in particular, Section 7.2 on vector bundles where the construction of the bundles TM and T^*M is worked out in detail. See also the references in Chapter 7.

When $M = \mathbb{R}^n$, observe that $T(M) = M \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, i.e., the bundle $T(M)$ is (globally) trivial.

Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds, we can define the function, $dh\colon T(M) \to T(N)$, (also denoted Th, or h_*, or Dh) by setting

$$dh(u) = dh_p(u), \quad \text{iff} \quad u \in T_p(M).$$

We leave the next proposition as an exercise to the reader (A proof can be found in Berger and Gostiaux [17]).

Proposition 3.10 *Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds M and N (with $k \geq 1$), the map $dh\colon T(M) \to T(N)$ is a C^{k-1}-map.*

We are now ready to define vector fields.

Definition 3.14 Let M be a C^{k+1} manifold, with $k \geq 1$. For any open subset, U of M, a *vector field on* U is any *section,* X, *of* $T(M)$ *over* U, i.e., any function, $X\colon U \to T(M)$, such that $\pi \circ X = \mathrm{id}_U$ (i.e., $X(p) \in T_p(M)$, for every $p \in U$). We also say that X is a *lifting of* U *into* $T(M)$. We say that X is a *C^k-vector field on* U iff X is a section over U and a C^k-map. The set of C^k-vector fields over U is denoted $\Gamma^{(k)}(U, T(M))$. Given a curve, $\gamma\colon [a, b] \to M$, a *vector field,* X, *along* γ is any section of $T(M)$ over γ, i.e., a C^k-function, $X\colon [a, b] \to T(M)$, such that $\pi \circ X = \gamma$. We also say that X *lifts* γ *into* $T(M)$.

The above definition gives a precise meaning to the idea that a C^k-vector field on M is an assignment, $p \mapsto X(p)$, of a tangent vector, $X(p) \in T_p(M)$, to a point, $p \in M$, so that $X(p)$ varies in a C^k-fashion in terms of p.

Clearly, $\Gamma^{(k)}(U, T(M))$ is a real vector space. For short, the space $\Gamma^{(k)}(M, T(M))$ is also denoted by $\Gamma^{(k)}(T(M))$ (or $\mathfrak{X}^{(k)}(M)$ or even $\Gamma(T(M))$ or $\mathfrak{X}(M)$).

Remark: We can also define a *C^j-vector field on* U as a section, X, over U which is a C^j-map, where $0 \leq j \leq k$. Then, we have the vector space, $\Gamma^{(j)}(U, T(M))$, *etc* .

If $M = \mathbb{R}^n$ and U is an open subset of M, then $T(M) = \mathbb{R}^n \times \mathbb{R}^n$ and a section of $T(M)$ over U is simply a function, X, such that

$$X(p) = (p, u), \quad \text{with} \quad u \in \mathbb{R}^n,$$

for all $p \in U$. In other words, X is defined by a function, $f\colon U \to \mathbb{R}^n$ (namely, $f(p) = u$). This corresponds to the "old" definition of a vector field in the more basic case where the manifold, M, is just $\mathbb{R}^n$.

Given any C^k-function, $f \in \mathcal{C}^k(U)$, and a vector field, $X \in \Gamma^{(k)}(U, T(M))$, we define the vector field, fX, by

$$(fX)(p) = f(p)X(p), \quad p \in U.$$

Obviously, $fX \in \Gamma^{(k)}(U, T(M))$, which shows that $\Gamma^{(k)}(U, T(M))$ is also a $\mathcal{C}^k(U)$-module. We also denote $X(p)$ by X_p. For any chart, (U, φ), on M it is easy to check that the map

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p, \quad p \in U,$$

is a C^k-vector field on U (with $1 \leq i \leq n$). This vector field is denoted $\left(\frac{\partial}{\partial x_i}\right)$ or $\frac{\partial}{\partial x_i}$.

Definition 3.15 Let M be a C^{k+1} manifold and let X be a C^k vector field on M. If U is any open subset of M and f is any function in $\mathcal{C}^k(U)$, then the *Lie derivative of f with respect to X*, denoted $X(f)$ or $L_X f$, is the function on U given by

$$X(f)(p) = X_p(f) = X_p(\mathbf{f}), \qquad p \in U.$$

Observe that

$$X(f)(p) = df_p(X_p),$$

where df_p is identified with the linear form in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}), \qquad v \in T_p M,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$ (see the discussion following Proposition 3.8). The Lie derivative, $L_X f$, is also denoted $X[f]$.

As a special case, when (U, φ) is a chart on M, the vector field, $\frac{\partial}{\partial x_i}$, just defined above induces the function

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f, \quad p \in U,$$

denoted $\frac{\partial}{\partial x_i}(f)$ or $\left(\frac{\partial}{\partial x_i}\right) f$.

It is easy to check that $X(f) \in \mathcal{C}^{k-1}(U)$. As a consequence, every vector field $X \in \Gamma^{(k)}(U, T(M))$ induces a linear map,

$$L_X \colon \mathcal{C}^k(U) \longrightarrow \mathcal{C}^{k-1}(U),$$

given by $f \mapsto X(f)$. It is immediate to check that L_X has the Leibnitz property, i.e.,

$$L_X(fg) = L_X(f)g + fL_X(g).$$

Linear maps with this property are called *derivations*. Thus, we see that every vector field induces some kind of differential operator, namely, a linear derivation. Unfortunately, not every linear derivation of the above type arises from a vector field, although this turns out to be true in the smooth case i.e., when $k = \infty$ (for a proof, see Gallot, Hulin and Lafontaine [60] or Lafontaine [92]).

In the rest of this section, unless stated otherwise, we assume that $k \geq 1$. The following easy proposition holds (c.f. Warner [145]):

Proposition 3.11 *Let X be a vector field on the C^{k+1}-manifold, M, of dimension n. Then, the following are equivalent:*

(a) X is C^k.

(b) If (U, φ) is a chart on M and if $f_1, \ldots, f_n$ are the functions on U uniquely defined by

$$X \restriction U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

then each f_i is a C^k-map.

(c) Whenever U is open in M and $f \in \mathcal{C}^k(U)$, then $X(f) \in \mathcal{C}^{k-1}(U)$.

Given any two C^k-vector field, X, Y, on M, for any function, $f \in \mathcal{C}^k(M)$, we defined above the function $X(f)$ and $Y(f)$. Thus, we can form $X(Y(f))$ (resp. $Y(X(f))$), which are in $\mathcal{C}^{k-2}(M)$. Unfortunately, even in the smooth case, there is generally no vector field, Z, such that

$$Z(f) = X(Y(f)), \quad \text{for all } f \in \mathcal{C}^k(M).$$

This is because $X(Y(f))$ (and $Y(X(f))$) involve second-order derivatives. However, if we consider $X(Y(f)) - Y(X(f))$, then second-order derivatives cancel out and there is a unique vector field inducing the above differential operator. Intuitively, $XY - YX$ measures the "failure of X and Y to commute".

Proposition 3.12 *Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, there is a unique C^{k-1}-vector field, $[X, Y]$, such that*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \textit{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

Proof. First we prove uniqueness. For this it is enough to prove that $[X, Y]$ is uniquely defined on $\mathcal{C}^k(U)$, for any chart, (U, φ). Over U, we know that

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i},$$

where $X_i, Y_i \in \mathcal{C}^k(U)$. Then, for any $f \in \mathcal{C}^k(M)$, we have

$$\begin{aligned} X(Y(f)) &= X\left(\sum_{j=1}^{n} Y_j \frac{\partial}{\partial x_j}(f)\right) = \sum_{i,j=1}^{n} X_i \frac{\partial}{\partial x_i}(Y_j) \frac{\partial}{\partial x_j}(f) + \sum_{i,j=1}^{n} X_i Y_j \frac{\partial^2}{\partial x_j \partial x_i}(f) \\ Y(X(f)) &= Y\left(\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}(f)\right) = \sum_{i,j=1}^{n} Y_j \frac{\partial}{\partial x_j}(X_i) \frac{\partial}{\partial x_i}(f) + \sum_{i,j=1}^{n} X_i Y_j \frac{\partial^2}{\partial x_i \partial x_j}(f). \end{aligned}$$

However, as $f \in \mathcal{C}^k(M)$, with $k \geq 2$, we have

$$\sum_{i,j=1}^{n} X_i Y_j \frac{\partial^2}{\partial x_j \partial x_i}(f) = \sum_{i,j=1}^{n} X_i Y_j \frac{\partial^2}{\partial x_i \partial x_j}(f),$$

and we deduce that

$$X(Y(f)) - Y(X(f)) = \sum_{i,j=1}^{n} \left(X_i \frac{\partial}{\partial x_i}(Y_j) - Y_i \frac{\partial}{\partial x_i}(X_j) \right) \frac{\partial}{\partial x_j}(f).$$

This proves that $[X, Y] = XY - YX$ is uniquely defined on U and that it is C^{k-1}. Thus, if $[X, Y]$ exists, it is unique.

To prove existence, we use the above expression to define $[X, Y]_U$, locally on U, for every chart, (U, φ). On any overlap, $U \cap V$, by the uniqueness property that we just proved, $[X, Y]_U$ and $[X, Y]_V$ must agree. But then, the $[X, Y]_U$ patch and yield a C^{k-1}-vector field defined on the whole of M. $\square$

Definition 3.16 Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, the *Lie bracket*, $[X, Y]$, of X and Y, is the C^{k-1} vector field defined so that

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \text{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

An an example, in $\mathbb{R}^3$, if X and Y are the two vector fields,

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y},$$

then

$$[X, Y] = -\frac{\partial}{\partial z}.$$

We also have the following simple proposition whose proof is left as an exercise (or, see Do Carmo [50]):

Proposition 3.13 *Given any C^{k+1}-manifold, M, of dimension n, for any C^k-vector fields, X, Y, Z, on M, for all $f, g \in \mathcal{C}^k(M)$, we have:*

(a) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity).*

(b) $[X, X] = 0$.

(c) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

(d) $[-, -]$ *is bilinear.*

As a consequence, for smooth manifolds ($k = \infty$), the space of vector fields, $\Gamma^{(\infty)}(T(M))$, is a vector space equipped with a bilinear operation, $[-,-]$, that satisfies the Jacobi identity. This makes $\Gamma^{(\infty)}(T(M))$ a *Lie algebra*.

Let $\varphi\colon M \to N$ be a diffeomorphism between two manifolds. Then, vector fields can be transported from N to M and conversely.

Definition 3.17 Let $\varphi\colon M \to N$ be a diffeomorphism between two C^{k+1} manifolds. For every C^k vector field, Y, on N, the *pull-back of* Y *along* φ is the vector field, φ^*Y, on M, given by

$$(\varphi^*Y)_p = d\varphi^{-1}_{\varphi(p)}(Y_{\varphi(p)}), \qquad p \in M.$$

For every C^k vector field, X, on M, the *push-forward of* X *along* φ is the vector field, φ_*X, on N, given by

$$\varphi_*X = (\varphi^{-1})^*X,$$

that is, for every $p \in M$,

$$(\varphi_*X)_{\varphi(p)} = d\varphi_p(X_p),$$

or equivalently,

$$(\varphi_*X)_q = d\varphi_{\varphi^{-1}(q)}(X_{\varphi^{-1}(q)}), \qquad q \in N.$$

It is not hard to check that

$$L_{\varphi_*X}f = L_X(f \circ \varphi) \circ \varphi^{-1},$$

for any function $f \in C^k(N)$.

One more notion will be needed when we deal with Lie algebras.

Definition 3.18 Let $\varphi\colon M \to N$ be a C^{k+1}-map of manifolds. If X is a C^k vector field on M and Y is a C^k vector field on N, we say that X *and* Y *are* φ*-related* iff

$$d\varphi \circ X = Y \circ \varphi.$$

The basic result about φ-related vector fields is:

Proposition 3.14 *Let $\varphi\colon M \to N$ be a C^{k+1}-map of manifolds, let X and Y be C^k vector fields on M and let X_1, Y_1 be C^k vector fields on N. If X is φ-related to X_1 and Y is φ-related to Y_1, then $[X, Y]$ is φ-related to $[X_1, Y_1]$.*

Proof. Basically, one needs to unwind the definitions, see Warner [145], Chapter 1. $\square$

3.4 Submanifolds, Immersions, Embeddings

Although the notion of submanifold is intuitively rather clear, technically, it is a bit tricky. In fact, the reader may have noticed that many different definitions appear in books and that it is not obvious at first glance that these definitions are equivalent. What is important is that a submanifold, N, of a given manifold, M, not only have the topology induced M but also that the charts of N be somewhow induced by those of M. (Recall that if X is a topological space and Y is a subset of X, then the *subspace topology on* Y or *topology induced by* X *on* Y has for its open sets all subsets of the form $Y \cap U$, where U is an arbitary open subset of X.).

Given m, n, with $0 \leq m \leq n$, we can view $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ using the inclusion

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{\underbrace{(0, \ldots, 0)}_{n-m}\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n, \quad (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m}).$$

Definition 3.19 Given a C^k-manifold, M, of dimension n, a subset, N, of M is an *m-dimensional submanifold of* M (where $0 \leq m \leq n$) iff for every point, $p \in N$, there is a chart, (U, φ), of M, with $p \in U$, so that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\}).$$

(We write $0_{n-m} = \underbrace{(0, \ldots, 0)}_{n-m}$.)

The subset, $U \cap N$, of Definition 3.19 is sometimes called a *slice* of (U, φ) and we say that (U, φ) is *adapted to* N (See O'Neill [117] or Warner [145]).

Other authors, including Warner [145], use the term submanifold in a broader sense than us and they use the word *embedded submanifold* for what is defined in Definition 3.19.

The following proposition has an almost trivial proof but it justifies the use of the word submanifold:

Proposition 3.15 *Given a C^k-manifold, M, of dimension n, for any submanifold, N, of M of dimension $m \leq n$, the family of pairs $(U \cap N, \varphi \restriction U \cap N)$, where (U, φ) ranges over the charts over any atlas for M, is an atlas for N, where N is given the subspace topology. Therefore, N inherits the structure of a C^k-manifold.*

In fact, every chart on N arises from a chart on M in the following precise sense:

Proposition 3.16 *Given a C^k-manifold, M, of dimension n and a submanifold, N, of M of dimension $m \leq n$, for any $p \in N$ and any chart, (W, η), of N at p, there is some chart, (U, φ), of M at p so that*

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\}) \quad \textit{and} \quad \varphi \restriction U \cap N = \eta \restriction U \cap N,$$

where $p \in U \cap N \subseteq W$.

Proof. See Berger and Gostiaux [17] (Chapter 2). □

It is also useful to define more general kinds of "submanifolds".

Definition 3.20 Let $\varphi\colon N \to M$ be a C^k-map of manifolds.

(a) The map φ is an *immersion* of N into M iff $d\varphi_p$ is injective for all $p \in N$.

(b) The set $\varphi(N)$ is an *immersed submanifold* of M iff φ is an injective immersion.

(c) The map φ is an *embedding* of N into M iff φ is an injective immersion such that the induced map, $N \longrightarrow \varphi(N)$, is a homeomorphism, where $\varphi(N)$ is given the subspace topology (equivalently, φ is an open map from N into $\varphi(N)$ with the subspace topology). We say that $\varphi(N)$ (with the subspace topology) is an *embedded submanifold* of M.

(d) The map φ is a *submersion* of N into M iff $d\varphi_p$ is surjective for all $p \in N$.

Again, we warn our readers that certain authors (such as Warner [145]) call $\varphi(N)$, in (b), a submanifold of M! We prefer the terminology *immersed submanifold*.

The notion of immersed submanifold arises naturally in the framewok of Lie groups. Indeed, the fundamental correspondence between Lie groups and Lie algebras involves Lie subgroups that are not necessarily closed. But, as we will see later, subgroups of Lie groups that are also submanifolds are always closed. It is thus necessary to have a more inclusive notion of submanifold for Lie groups and the concept of immersed submanifold is just what's needed.

Immersions of $\mathbb{R}$ into $\mathbb{R}^3$ are parametric curves and immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$ are parametric surfaces. These have been extensively studied, for example, see DoCarmo [49], Berger and Gostiaux [17] or Gallier [58].

Immersions (i.e., subsets of the form $\varphi(N)$, where N is an immersion) are generally neither injective immersions (i.e., subsets of the form $\varphi(N)$, where N is an injective immersion) nor embeddings (or submanifolds). For example, immersions can have self-intersections, as the plane curve (nodal cubic): $x = t^2 - 1; y = t(t^2 - 1)$.

Injective immersions are generally not embeddings (or submanifolds) because $\varphi(N)$ may not be homeomorphic to N. An example is given by the Lemniscate of Bernoulli, an injective immersion of $\mathbb{R}$ into $\mathbb{R}^2$:

$$\begin{aligned} x &= \frac{t(1+t^2)}{1+t^4}, \\ y &= \frac{t(1-t^2)}{1+t^4}. \end{aligned}$$

Another interesting example is the immersion of $\mathbb{R}$ into the 2-torus, $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$, given by

$$t \mapsto (\cos t, \sin t, \cos ct, \sin ct),$$

where $c \in \mathbb{R}$. One can show that the image of $\mathbb{R}$ under this immersion is closed in T^2 iff c is rational. Moreover, the image of this immersion is dense in T^2 but not closed iff c is irrational. The above example can be adapted to the torus in $\mathbb{R}^3$: One can show that the immersion given by

$$t \mapsto ((2 + \cos t)\cos(\sqrt{2}\,t), (2 + \cos t)\sin(\sqrt{2}\,t), \sin t),$$

is dense but not closed in the torus (in $\mathbb{R}^3$) given by

$$(s, t) \mapsto ((2 + \cos s)\cos t, (2 + \cos s)\sin t, \sin s),$$

where $s, t \in \mathbb{R}$.

There is, however, a close relationship between submanifolds and embeddings.

Proposition 3.17 *If N is a submanifold of M, then the inclusion map, $j\colon N \to M$, is an embedding. Conversely, if $\varphi\colon N \to M$ is an embedding, then $\varphi(N)$ with the subspace topology is a submanifold of M and φ is a diffeomorphism between N and $\varphi(N)$.*

Proof. See O'Neill [117] (Chapter 1) or Berger and Gostiaux [17] (Chapter 2). □

In summary, embedded submanifolds and (our) submanifolds coincide. Some authors refer to spaces of the form $\varphi(N)$, where φ is an injective immersion, as *immersed submanifolds* and we have adopted this terminology. However, in general, an immersed submanifold is *not* a submanifold. One case where this holds is when N is compact, since then, a bijective continuous map is a homeomorphism. For yet a notion of submanifold intermediate between immersed submanifolds and (our) submanifolds, see Sharpe [137] (Chapter 1).

Our next goal is to review and promote to manifolds some standard results about ordinary differential equations.

3.5 Integral Curves, Flow of a Vector Field, One-Parameter Groups of Diffeomorphisms

We begin with integral curves and (local) flows of vector fields on a manifold.

Definition 3.21 Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. An *integral curve (or trajectory) for X with initial condition p_0* is a C^{k-1}-curve, $\gamma\colon I \to M$, so that

$$\dot{\gamma}(t) = X(\gamma(t)), \quad \text{for all } t \in I \quad \text{and} \quad \gamma(0) = p_0,$$

where $I =]a, b[\subseteq \mathbb{R}$ is an open interval containing 0.

What definition 3.21 says is that an integral curve, γ, with initial condition p_0 is a curve on the manifold M passing through p_0 and such that, for every point $p = \gamma(t)$ on this curve, the tangent vector to this curve at p, i.e., $\dot{\gamma}(t)$, coincides with the value, $X(p)$, of the vector field X at p.

Given a vector field, X, as above, and a point $p_0 \in M$, is there an integral curve through p_0? Is such a curve unique? If so, how large is the open interval I? We provide some answers to the above questions below.

Definition 3.22 Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. A *local flow for X at p_0* is a map,

$$\varphi\colon J \times U \to M,$$

where $J \subseteq \mathbb{R}$ is an open interval containing 0 and U is an open subset of M containing p_0, so that for every $p \in U$, the curve $t \mapsto \varphi(t, p)$ is an integral curve of X with initial condition p.

Thus, a local flow for X is a family of integral curves for all points in some small open set around p_0 such that these curves all have the same domain, J, independently of the initial condition, $p \in U$.

The following theorem is the main existence theorem of local flows. This is a promoted version of a similar theorem in the classical theory of ODE's in the case where M is an open subset of $\mathbb{R}^n$. For a full account of this theory, see Lang [95] or Berger and Gostiaux [17].

Theorem 3.18 *(Existence of a local flow) Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. There is an open interval, $J \subseteq \mathbb{R}$, containing 0 and an open subset, $U \subseteq M$, containing p_0, so that there is a* **unique** *local flow, $\varphi\colon J \times U \to M$, for X at p_0. Furthermore, φ is C^{k-1}.*

Theorem 3.18 holds under more general hypotheses, namely, when the vector field satisfies some *Lipschitz* condition, see Lang [95] or Berger and Gostiaux [17].

Now, we know that for any initial condition, p_0, there is some integral curve through p_0. However, there could be two (or more) integral curves $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ with initial condition p_0. This leads to the natural question: How do γ_1 and γ_2 differ on $I_1 \cap I_2$? The next proposition shows they don't!

Proposition 3.19 *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. If $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ are any two integral curves both with initial condition p_0, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.*

Proof. Let $Q = \{t \in I_1 \cap I_2 \mid \gamma_1(t) = \gamma_2(t)\}$. Since $\gamma_1(0) = \gamma_2(0) = p_0$, the set Q is nonempty. If we show that Q is both closed and open in $I_1 \cap I_2$, as $I_1 \cap I_2$ is connected since it is an open interval of $\mathbb{R}$, we will be able to conclude that $Q = I_1 \cap I_2$.

Since by definition, a manifold is Hausdorff, it is a standard fact in topology that the diagonal, $\Delta = \{(p, p) \mid p \in M\} \subseteq M \times M$, is closed, and since

$$Q = I_1 \cap I_2 \cap (\gamma_1, \gamma_2)^{-1}(\Delta)$$

and γ_1 and γ_2 are continuous, we see that Q is closed in $I_1 \cap I_2$.

Pick any $u \in Q$ and consider the curves β_1 and β_2 given by

$$\beta_1(t) = \gamma_1(t + u) \quad \text{and} \quad \beta_2(t) = \gamma_2(t + u),$$

where $t \in I_1 - u$ in the first case and $t \in I_2 - u$ in the second. (Here, if $I = \,]a, b[\,$, we have $I - u = \,]a - u, b - u[\,$.) Observe that

$$\dot{\beta}_1(t) = \dot{\gamma}_1(t + u) = X(\gamma_1(t + u)) = X(\beta_1(t))$$

and similarly, $\dot{\beta}_2(t) = X(\beta_2(t))$. We also have

$$\beta_1(0) = \gamma_1(u) = \gamma_2(u) = \beta_2(0) = q,$$

since $u \in Q$ (where $\gamma_1(u) = \gamma_2(u)$). Thus, $\beta_1\colon (I_1 - u) \to M$ and $\beta_2\colon (I_2 - u) \to M$ are two integral curves with the same initial condition, q. By Theorem 3.18, the uniqueness of local flow implies that there is some open interval, $\widetilde{I} \subseteq I_1 \cap I_2 - u$, such that $\beta_1 = \beta_2$ on $\widetilde{I}$. Consequently, γ_1 and γ_2 agree on $\widetilde{I} + u$, an open subset of Q, proving that Q is indeed open in $I_1 \cap I_2$. □

Proposition 3.19 implies the important fact that there is a *unique maximal* integral curve with initial condition p. Indeed, if $\{\gamma_j\colon I_j \to M\}_{j \in K}$ is the family of all integral curves with initial condition p (for some big index set, K), if we let $I(p) = \bigcup_{j \in K} I_j$, we can define a curve, $\gamma_p\colon I(p) \to M$, so that

$$\gamma_p(t) = \gamma_j(t), \quad \text{if} \quad t \in I_j.$$

Since γ_j and γ_l agree on $I_j \cap I_l$ for all $j, l \in K$, the curve γ_p is indeed well defined and it is clearly an integral curve with initial condition p with the largest possible domain (the open interval, $I(p)$). The curve γ_p is called the *maximal integral curve with initial condition p* and it is also denoted by $\gamma(p, t)$. Note that Proposition 3.19 implies that any two distinct integral curves are disjoint, i.e., do not intersect each other.

Consider the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

If we write $\gamma(t) = (x(t), y(t))$, the differential equation, $\dot{\gamma}(t) = X(\gamma(t))$, is expressed by

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t), \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we write $X = \binom{x}{y}$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the above equation is written as

$$X' = AX.$$

Now, as

$$e^{tA} = I + \frac{A}{1!}t + \frac{A^2}{2!}t^2 + \cdots + \frac{A^n}{n!}t^n + \cdots,$$

we get

$$\frac{d}{dt}(e^{tA}) = A + \frac{A^2}{1!}t + \frac{A^3}{2!}t^2 + \cdots + \frac{A^n}{(n-1)!}t^{n-1} + \cdots = Ae^{tA},$$

so we see that $e^{tA}p$ is a solution of the ODE $X' = AX$ with initial condition $X = p$, and by uniqueness, $X = e^{tA}p$ is *the* solution of our ODE starting at $X = p$. Thus, our integral curve, γ_p, through $p = \binom{x_0}{y_0}$ is the circle given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Observe that $I(p) = \mathbb{R}$, for every $p \in \mathbb{R}^2$.

The following interesting question now arises: Given any $p_0 \in M$, if $\gamma_{p_0} \colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0 and, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$, then there is a maximal integral curve, $\gamma_{p_1} \colon I(p_1) \to M$, with initial condition p_1; what is the relationship between γ_{p_0} and γ_{p_1}, if any? The answer is given by

Proposition 3.20 *Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. If $\gamma_{p_0} \colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$ and $\gamma_{p_1} \colon I(p_1) \to M$ is the maximal integral curve with initial condition p_1, then*

$$I(p_1) = I(p_0) - t_1 \quad \text{and} \quad \gamma_{p_1}(t) = \gamma_{\gamma_{p_0}(t_1)}(t) = \gamma_{p_0}(t + t_1), \quad \text{for all } t \in I(p_0) - t_1.$$

Proof. Let $\gamma(t)$ be the curve given by

$$\gamma(t) = \gamma_{p_0}(t + t_1), \quad \text{for all } \ t \in I(p_0) - t_1.$$

Clearly, γ is defined on $I(p_0) - t_1$ and

$$\dot{\gamma}(t) = \dot{\gamma}_{p_0}(t + t_1) = X(\gamma_{p_0}(t + t_1)) = X(\gamma(t))$$

and $\gamma(0) = \gamma_{p_0}(t_1) = p_1$. Thus, γ is an integal curve defined on $I(p_0) - t_1$ with initial condition p_1. If γ was defined on an interval, $\widetilde{I} \supseteq I(p_0) - t_1$ with $\widetilde{I} \neq I(p_0) - t_1$, then γ_{p_0} would be defined on $\widetilde{I} + t_1 \supset I(p_0)$, an interval strictly bigger than $I(p_0)$, contradicting the maximality of $I(p_0)$. Therefore, $I(p_0) - t_1 = I(p_1)$. $\square$

It is useful to restate Proposition 3.20 by changing point of view. So far, we have been focusing on integral curves, i.e., given any $p_0 \in M$, we let t vary in $I(p_0)$ and get an integral curve, γ_{p_0}, with domain $I(p_0)$.

Instead of holding $p_0 \in M$ fixed, we can hold $t \in \mathbb{R}$ fixed and consider the set

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\},$$

i.e., the set of points such that it is possible to "travel for t units of time from p" along the maximal integral curve, γ_p, with initial condition p (It is possible that $\mathcal{D}_t(X) = \emptyset$). By definition, if $\mathcal{D}_t(X) \neq \emptyset$, the point $\gamma_p(t)$ is well defined, and so, we obtain a map, $\Phi_t^X : \mathcal{D}_t(X) \to M$, with domain $\mathcal{D}_t(X)$, given by

$$\Phi_t^X(p) = \gamma_p(t).$$

The above suggests the following definition:

Definition 3.23 Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. For any $t \in \mathbb{R}$, let

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\} \quad \text{and} \quad \mathcal{D}(X) = \{(t, p) \in \mathbb{R} \times M \mid t \in I(p)\}$$

and let $\Phi^X : \mathcal{D}(X) \to M$ be the map given by

$$\Phi^X(t, p) = \gamma_p(t).$$

The map Φ^X is called the *(global) flow of* X and $\mathcal{D}(X)$ is called its *domain of definition*. For any $t \in \mathbb{R}$ such that $\mathcal{D}_t(X) \neq \emptyset$, the map, $p \in \mathcal{D}_t(X) \mapsto \Phi^X(t, p) = \gamma_p(t)$, is denoted by Φ_t^X (i.e., $\Phi_t^X(p) = \Phi^X(t, p) = \gamma_p(t)$).

Observe that

$$\mathcal{D}(X) = \bigcup_{p \in M} (I(p) \times \{p\}).$$

Also, using the Φ_t^X notation, the property of Proposition 3.20 reads

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X, \qquad (*)$$

whenever both sides of the equation make sense. Indeed, the above says

$$\Phi_s^X(\Phi_t^X(p)) = \Phi_s^X(\gamma_p(t)) = \gamma_{\gamma_p(t)}(s) = \gamma_p(s+t) = \Phi_{s+t}^X(p).$$

Using the above property, we can easily show that the Φ_t^X are invertible. In fact, the inverse of Φ_t^X is Φ_{-t}^X. First, note that

$$\mathcal{D}_0(X) = M \quad \text{and} \quad \Phi_0^X = \mathrm{id},$$

because, by definition, $\Phi_0^X(p) = \gamma_p(0) = p$, for every $p \in M$. Then, $(*)$ implies that

$$\Phi_t^X \circ \Phi_{-t}^X = \Phi_{t+-t}^X = \Phi_0^X = \mathrm{id},$$

which shows that $\Phi_t^X \colon \mathcal{D}_t(X) \to \mathcal{D}_{-t}(X)$ and $\Phi_{-t}^X \colon \mathcal{D}_{-t}(X) \to \mathcal{D}_t(X)$ are inverse of each other. Moreover, each Φ_t^X is a C^{k-1}-diffeomorphism. We summarize in the following proposition some additional properties of the domains $\mathcal{D}(X)$, $\mathcal{D}_t(X)$ and the maps Φ_t^X (for a proof, see Lang [95] or Warner [145]):

Theorem 3.21 *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. The following properties hold:*

(a) For every $t \in \mathbb{R}$, if $\mathcal{D}_t(X) \neq \emptyset$, then $\mathcal{D}_t(X)$ is open (this is trivially true if $\mathcal{D}_t(X) = \emptyset$).

(b) The domain, $\mathcal{D}(X)$, of the flow, Φ^X, is open and the flow is a C^{k-1} map, $\Phi^X \colon \mathcal{D}(X) \to M$.

(c) Each $\Phi_t^X \colon \mathcal{D}_t(X) \to \mathcal{D}_{-t}(X)$ is a C^{k-1}-diffeomorphism with inverse Φ_{-t}^X.

(d) For all $s, t \in \mathbb{R}$, the domain of definition of $\Phi_s^X \circ \Phi_t^X$ is contained but generally not equal to $\mathcal{D}_{s+t}(X)$. However, $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X) = \mathcal{D}_{s+t}(X)$ if s and t have the same sign. Moreover, on $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X)$, we have

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X.$$

Remarks:

(1) We may omit the superscript, X, and write Φ instead of Φ^X if no confusion arises.

(2) The reason for using the terminology flow in referring to the map Φ^X can be clarified as follows: For any t such that $\mathcal{D}_t(X) \neq \emptyset$, every integral curve, γ_p, with initial condition $p \in \mathcal{D}_t(X)$, is defined on some open interval containing $[0, t]$, and we can picture these curves as "flow lines" along which the points p flow (travel) for a time interval t. Then, $\Phi^X(t, p)$ is the point reached by "flowing" for the amount of time t on the integral curve γ_p (through p) starting from p. Intuitively, we can imagine the flow of a fluid through M, and the vector field X is the field of velocities of the flowing particles.

Given a vector field, X, as above, it may happen that $\mathcal{D}_t(X) = M$, for all $t \in \mathbb{R}$. In this case, namely, when $\mathcal{D}(X) = \mathbb{R} \times M$, we say that the vector field X is *complete*. Then, the Φ_t^X are diffeomorphisms of M and they form a group. The family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ a called a 1-*parameter group of* X. In this case, Φ^X induces a group homomorphism, $(\mathbb{R}, +) \longrightarrow \mathrm{Diff}(M)$, from the additive group $\mathbb{R}$ to the group of C^{k-1}-diffeomorphisms of M.

By abuse of language, even when it is **not** the case that $\mathcal{D}_t(X) = M$ for all t, the family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ is called a *local* 1-*parameter group generated by* X, even though it is **not** a group, because the composition $\Phi_s^X \circ \Phi_t^X$ may not be defined.

If we go back to the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

since the integral curve, $\gamma_p(t)$, through $p = \binom{x_0}{x_0}$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

the global flow associated with X is given by

$$\Phi^X(t, p) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} p,$$

and each diffeomorphism, Φ_t^X, is the rotation,

$$\Phi_t^X = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The 1-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is the group of rotations in the plane, $\mathbf{SO}(2)$.

More generally, if B is an $n \times n$ invertible matrix that has a real logarithm, A (that is, if $e^A = B$), then the matrix A defines a vector field, X, in $\mathbb{R}^n$, with

$$X = \sum_{i,j=1}^{n} (a_{ij}x_j)\frac{\partial}{\partial x_i},$$

whose integral curves are of the form,

$$\gamma_p(t) = e^{tA}p.$$

The one-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is given by $\{e^{tA}\}_{t\in\mathbb{R}}$.

When M is compact, it turns out that every vector field is complete, a nice and useful fact.

Proposition 3.22 *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. If M is compact, then X is complete, i.e., $\mathcal{D}(X) = \mathbb{R} \times M$. Moreover, the map $t \mapsto \Phi_t^X$ is a homomorphism from the additive group $\mathbb{R}$ to the group,* $\mathrm{Diff}(M)$, *of (C^{k-1}) diffeomorphisms of M.*

Proof. Pick any $p \in M$. By Theorem 3.18, there is a local flow, $\varphi_p \colon J(p) \times U(p) \to M$, where $J(p) \subseteq \mathbb{R}$ is an open interval containing 0 and $U(p)$ is an open subset of M containing p, so that for all $q \in U(p)$, the map $t \mapsto \varphi(t, q)$ is an integral curve with initial condition q (where $t \in J(p)$). Thus, we have $J(p) \times U(p) \subseteq \mathcal{D}(X)$. Now, the $U(p)$'s form an open cover of M and since M is compact, we can extract a finite subcover, $\bigcup_{q \in F} U(q) = M$, for some finite subset, $F \subseteq M$. But then, we can find $\epsilon > 0$ so that $]-\epsilon, +\epsilon[\, \subseteq J(q)$, for all $q \in F$ and for all $t \in\,]-\epsilon, +\epsilon[$ and, for all $p \in M$, if γ_p is the maximal integral curve with initial condition p, then $]-\epsilon, +\epsilon[\, \subseteq I(p)$.

For any $t \in\,]-\epsilon, +\epsilon[$, consider the integral curve, $\gamma_{\gamma_p(t)}$, with initial condition $\gamma_p(t)$. This curve is well defined for all $t \in\,]-\epsilon, +\epsilon[$, and we have

$$\gamma_{\gamma_p(t)}(t) = \gamma_p(t + t) = \gamma_p(2t),$$

which shows that γ_p is in fact defined for all $t \in\,]-2\epsilon, +2\epsilon[$. By induction, we see that

$$]-2^n\epsilon, +2^n\epsilon[\, \subseteq I(p),$$

for all $n \geq 0$, which proves that $I(p) = \mathbb{R}$. As this holds for all $p \in M$, we conclude that $\mathcal{D}(X) = \mathbb{R} \times M$. $\square$

Remarks:

(1) The proof of Proposition 3.22 also applies when X is a vector field with compact support (this means that the closure of the set $\{p \in M \mid X(p) \neq 0\}$ is compact).

(2) If $\varphi \colon M \to N$ is a diffeomorphism and X is a vector field on M, then it can be shown that the local 1-parameter group associated with the vector field, $\varphi_* X$, is

$$(\varphi \circ \Phi_t^X \circ \varphi^{-1}).$$

A point $p \in M$ where a vector field vanishes, i.e., $X(p) = 0$, is called a *critical point of X*. Critical points play a major role in the study of vector fields, in differential topology (e.g., the celebrated Poincaré–Hopf index theorem) and especially in Morse theory, but we won't go into this here (curious readers should consult Milnor [105], Guillemin and Pollack [69] or DoCarmo [49], which contains an informal but very clear presentation of the Poincaré–Hopf index theorem). Another famous theorem about vector fields says that every smooth vector field on a sphere of even dimension (S^{2n}) must vanish in at least one point (the so-called "hairy-ball theorem". On S^2, it says that you can't comb your hair without having a singularity somewhere. Try it, it's true!).

Let us just observe that if an integral curve, γ, passes through a critical point, p, then γ is reduced to the point p, i.e., $\gamma(t) = p$, for all t. Indeed, such a curve is an integral curve with initial condition p. By the uniqueness property, it is the only one. Then, we see that if a maximal integral curve is defined on the whole of $\mathbb{R}$, either it is injective (it has no self-intersection), or it is simply periodic (i.e., there is some $T > 0$ so that $\gamma(t + T) = \gamma(t)$, for all $t \in \mathbb{R}$ and γ is injective on $[0, T[$), or it is reduced to a single point.

We conclude this section with the definition of the Lie derivative of a vector field with respect to another vector field.

Say we have two vector fields X and Y on M. For any $p \in M$, we can flow along the integral curve of X with initial condition p to $\Phi_t(p)$ (for t small enough) and then evaluate Y there, getting $Y(\Phi_t(p))$. Now, this vector belongs to the tangent space $T_{\Phi_t(p)}(M)$, but $Y(p) \in T_p(M)$. So to "compare" $Y(\Phi_t(p))$ and $Y(p)$, we bring back $Y(\Phi_t(p))$ to $T_p(M)$ by applying the tangent map, $d\Phi_{-t}$, at $\Phi_t(p)$, to $Y(\Phi_t(p))$ (Note that to alleviate the notation, we use the slight abuse of notation $d\Phi_{-t}$ instead of $d(\Phi_{-t})_{\Phi_t(p)}$.) Then, we can form the difference $d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)$, divide by t and consider the limit as t goes to 0.

Definition 3.24 Let M be a C^{k+1} manifold. Given any two C^k vector fields, X and Y on M, for every $p \in M$, the *Lie derivative of* Y *with respect to* X *at* p, denoted $(L_X\, Y)_p$, is given by

$$(L_X\, Y)_p = \lim_{t \longrightarrow 0} \frac{d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)}{t} = \frac{d}{dt}\left(d\Phi_{-t}(Y(\Phi_t(p)))\right)\bigg|_{t=0}.$$

It can be shown that $(L_X\, Y)_p$ is our old friend, the Lie bracket, i.e.,

$$(L_X\, Y)_p = [X, Y]_p.$$

(For a proof, see Warner [145] or O'Neill [117]).

In terms of Definition 3.17, observe that

$$(L_X\, Y)_p = \lim_{t \longrightarrow 0} \frac{\big((\Phi_{-t})_* Y\big)(p) - Y(p)}{t} = \lim_{t \longrightarrow 0} \frac{\big(\Phi_t^* Y\big)(p) - Y(p)}{t} = \frac{d}{dt}\big(\Phi_t^* Y\big)(p)\bigg|_{t=0},$$

since $(\Phi_{-t})^{-1} = \Phi_t$.

3.6 Partitions of Unity

To study manifolds, it is often necessary to construct various objects such as functions, vector fields, Riemannian metrics, volume forms, etc., by gluing together items constructed on the domains of charts. Partitions of unity are a crucial technical tool in this gluing process.

The first step is to define "bump functions" (also called plateau functions). For any $r > 0$, we denote by $B(r)$ the open ball

$$B(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\},$$

and by $\overline{B(r)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq r\}$, its closure.

Proposition 3.23 *There is a smooth function, $b \colon \mathbb{R}^n \to \mathbb{R}$, so that*

$$b(x) = \begin{cases} 1 & \text{if } x \in \overline{B(1)} \\ 0 & \text{if } x \in \mathbb{R}^n - B(2). \end{cases}$$

Proof. There are many ways to construct such a function. We can proceed as follows: Consider the function, $h \colon \mathbb{R} \to \mathbb{R}$, given by

$$h(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

It is easy to show that h is C^∞ (but **not** analytic!). Then, define $b \colon \mathbb{R}^n \to \mathbb{R}$, by

$$b(x_1, \ldots, x_n) = \frac{h(4 - x_1^2 - \cdots - x_n^2)}{h(4 - x_1^2 - \cdots - x_n^2) + h(x_1^2 + \cdots + x_n^2 - 1)}.$$

It is immediately verified that b satisfies the required conditions. □

Given a topological space, X, for any function, $f \colon X \to \mathbb{R}$, the *support of f*, denoted $\mathrm{supp}\, f$, is the closed set,

$$\mathrm{supp}\, f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Proposition 3.23 yields the following useful technical result:

Proposition 3.24 *Let M be a smooth manifold. For any open subset, $U \subseteq M$, any $p \in U$ and any smooth function, $f \colon U \to \mathbb{R}$, there exist an open subset, V, with $p \in V$ and a smooth function, $\widetilde{f} \colon M \to \mathbb{R}$, defined on the whole of M, so that $\overline{V}$ is compact,*

$$\overline{V} \subseteq U, \qquad \mathrm{supp}\, \widetilde{f} \subseteq U$$

and

$$\widetilde{f}(q) = f(q), \qquad \textit{for all} \quad q \in \overline{V}.$$

Proof. Using a scaling function, it is easy to find a chart, (W, φ) at p, so that $W \subseteq U$, $B(3) \subseteq \varphi(W)$ and $\varphi(p) = 0$. Let $\widetilde{b} = b \circ \varphi$, where b is the function given by Proposition 3.23. Then, $\widetilde{b}$ is a smooth function on W with support $\varphi^{-1}(\overline{B(2)}) \subseteq W$. We can extend $\widetilde{b}$ outside W, by setting it to be 0 and we get a smooth function on the whole M. If we let

$V = \varphi^{-1}(B(1))$, then V is an open subset around p, $\overline{V} = \varphi^{-1}(\overline{B(1)}) \subseteq W$ is compact and, clearly, $\widetilde{b} = 1$ on $\overline{V}$. Therefore, if we set

$$\widetilde{f}(q) = \begin{cases} \widetilde{b}(q)f(q) & \text{if } q \in W \\ 0 & \text{if } q \in M - W, \end{cases}$$

we see that $\widetilde{f}$ satisfies the required properties. □

If X is a (Hausdorff) topological space, a family, $\{U_\alpha\}_{\alpha \in I}$, of subsets U_α of X is a *cover* (or *covering*) of X iff $X = \bigcup_{\alpha \in I} U_\alpha$. A cover, $\{U_\alpha\}_{\alpha \in I}$, such that each U_α is open is an *open cover*. If $\{U_\alpha\}_{\alpha \in I}$ is a cover of X, for any subset, $J \subseteq I$, the subfamily $\{U_\alpha\}_{\alpha \in J}$ is a *subcover* of $\{U_\alpha\}_{\alpha \in I}$ if $X = \bigcup_{\alpha \in J} U_\alpha$, i.e., $\{U_\alpha\}_{\alpha \in J}$ is still a cover of X. Given two covers, $\{U_\alpha\}_{\alpha \in I}$ and $\{V_\beta\}_{\beta \in J}$, we say that $\{U_\alpha\}_{\alpha \in I}$ is a *refinement* of $\{V_\beta\}_{\beta \in J}$ iff there is a function, $h \colon I \to J$, so that $U_\alpha \subseteq V_{h(\alpha)}$, for all $\alpha \in I$.

A cover, $\{U_\alpha\}_{\alpha \in I}$, is *locally finite* iff for every point, $p \in X$, there is some open subset, U, with $p \in U$, so that $U \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in I$. A space, X, is *paracompact* iff every open cover has an open locally finite refinement.

Remark: Recall that a space, X, is *compact* iff it is Hausdorff and if every open cover has a *finite* subcover. Thus, the notion of paracompactess (due to Jean Dieudonné) is a generalization of the notion of compactness.

Recall that a topological space, X, is *second-countable* if it has a countable basis, i.e., if there is a countable family of open subsets, $\{U_i\}_{i \geq 1}$, so that every open subset of X is the union of some of the U_i's. A topological space, X, if *locally compact* iff it is Hausdorff and for every $a \in X$, there is some compact subset, K, and some open subset, U, with $a \in U$ and $U \subseteq K$. As we will see shortly, every locally compact and second-countable topological space is paracompact.

It is important to observe that every manifold (even not second-countable) is locally compact. Indeed, for every $p \in M$, if we pick a chart, (U, φ), around p, then $\varphi(U) = \Omega$ for some open $\Omega \subseteq \mathbb{R}^n$ ($n = \dim M$). So, we can pick a small closed ball, $\overline{B(q, \epsilon)} \subseteq \Omega$, of center $q = \varphi(p)$ and radius ϵ, and as φ is a homeomorphism, we see that

$$p \in \varphi^{-1}(B(q, \epsilon/2)) \subseteq \varphi^{-1}(\overline{B(q, \epsilon)}),$$

where $\varphi^{-1}(\overline{B(q, \epsilon)})$ is compact and $\varphi^{-1}(B(q, \epsilon/2))$ is open.

Finally, we define partitions of unity.

Definition 3.25 Let M be a (smooth) manifold. A *partition of unity on M* is a family, $\{f_i\}_{i \in I}$, of smooth functions on M (the index set I may be uncountable) such that

(a) The family of supports, $\{\operatorname{supp} f_i\}_{i \in I}$, is locally finite.

(b) For all $i \in I$ and all $p \in M$, we have $0 \leq f_i(p) \leq 1$, and

$$\sum_{i \in I} f_i(p) = 1, \quad \text{for every } p \in M.$$

If $\{U_\alpha\}_{\alpha \in J}$ is a cover of M, we say that the partition of unity $\{f_i\}_{i \in I}$ is *subordinate* to the cover $\{U_\alpha\}_{\alpha \in J}$ if $\{\text{supp}\, f_i\}_{i \in I}$ is a refinement of $\{U_\alpha\}_{\alpha \in J}$. When $I = J$ and $\text{supp}\, f_i \subseteq U_i$, we say that $\{f_i\}_{i \in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha \in I}$ *with the same index set as the partition of unity.*

In Definition 3.25, by (a), for every $p \in M$, there is some open set, U, with $p \in U$ and U meets only finitely many of the supports, $\text{supp}\, f_i$. So, $f_i(p) \neq 0$ for only finitely many $i \in I$ and the infinite sum $\sum_{i \in I} f_i(p)$ is well defined.

Proposition 3.25 *Let X be a topological space which is second-countable and locally compact (thus, also Hausdorff). Then, X is paracompact. Moreover, every open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

Proof. The proof is quite technical, but since this is an important result, we reproduce Warner's proof for the reader's convenience (Warner [145], Lemma 1.9).

The first step is to construct a sequence of open sets, G_i, such that

1. $\overline{G}_i$ is compact,
2. $\overline{G}_i \subseteq G_{i+1}$,
3. $X = \bigcup_{i=1}^{\infty} G_i$.

As M is second-countable, there is a countable basis of open sets, $\{U_i\}_{i \geq 1}$, for M. Since M is locally compact, we can find a subfamily of $\{U_i\}_{i \geq 1}$ consisting of open sets with compact closures such that this subfamily is also a basis of M. Therefore, we may assume that we start with a countable basis, $\{U_i\}_{i \geq 1}$, of open sets with compact closures. Set $G_1 = U_1$ and assume inductively that

$$G_k = U_1 \cup \cdots \cup U_{j_k}.$$

Since $\overline{G}_k$ is compact, it is covered by finitely many of the U_j's. So, let j_{k+1} be the smallest integer greater than j_k so that

$$\overline{G}_k \subseteq U_1 \cup \cdots \cup U_{j_{k+1}}$$

and set

$$G_{k+1} = U_1 \cup \cdots \cup U_{j_{k+1}}.$$

Obviously, the family $\{G_i\}_{i \geq 1}$ satisfies (1)–(3).

Now, let $\{U_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of M. For any $i \geq 3$, the set $\overline{G}_i - G_{i-1}$ is compact and contained in the open $G_{i+1} - \overline{G}_{i-2}$. For each $i \geq 3$, choose a finite subcover of the open cover $\{U_\alpha \cap (G_{i+1} - \overline{G}_{i-2})\}_{\alpha \in I}$ of $\overline{G}_i - G_{i-1}$, and choose a finite subcover of the open cover $\{U_\alpha \cap G_3\}_{\alpha\in I}$ of the compact set $\overline{G}_2$. We leave it to the reader to check that this family of open sets is indeed a countable, locally finite refinement of the original open cover $\{U_\alpha\}_{\alpha\in I}$ and consists of open sets with compact closures. $\square$

Remarks:

1. Proposition 3.25 implies that a second-countable, locally compact (Hausdorff) topological space is the union of countably many compact subsets. Thus, X is *countable at infinity*, a notion that we already encountered in Proposition 2.23 and Theorem 2.26. The reason for this odd terminology is that in the Alexandroff one-point compactification of X, the family of open subsets containing the point at infinity (ω) has a countable basis of open sets. (The open subsets containing ω are of the form $(M - K) \cup \{\omega\}$, where K is compact.)

2. A manifold that is countable at infinity has a countable open cover by domains of charts. This is because, if $M = \bigcup_{i\geq 1} K_i$, where the $K_i \subseteq M$ are compact, then for any open cover of M by domains of charts, for every K_i, we can extract a finite subcover, and the union of these finite subcovers is a countable open cover of M by domains of charts. But then, since for every chart, (U_i, φ_i), the map φ_i is a homeomorphism onto some open subset of $\mathbb{R}^n$, which is second-countable, so we deduce easily that M is second-countable. Thus, for manifolds, second-countable is equivalent to countable at infinity.

We can now prove the main theorem stating the existence of partitions of unity. Recall that we are assuming that our manifolds are Hausdorff and second-countable.

Theorem 3.26 *Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in I}$ be an open cover for M. Then, there is a countable partition of unity, $\{f_i\}_{i\geq 1}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in I}$ and the support,* $\mathrm{supp}\, f_i$*, of each f_i is compact. If one does not require compact supports, then there is a partition of unity, $\{f_\alpha\}_{\alpha\in I}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in I}$ with at most countably many of the f_α not identically zero. (In the second case,* $\mathrm{supp}\, f_\alpha \subseteq U_\alpha$*.)*

Proof. Again, we reproduce Warner's proof (Warner [145], Theorem 1.11). As our manifolds are second-countable, Hausdorff and locally compact, from the proof of Proposition 3.25, we have the sequence of open subsets, $\{G_i\}_{i\geq 1}$ and we set $G_0 = \emptyset$. For any $p \in M$, let i_p be the largest integer such that $p \in M - \overline{G}_{i_p}$. Choose an α_p such that $p \in U_{\alpha_p}$; we can find a chart, (U, φ), centered at p such that $U \subseteq U_{\alpha_p} \cap (G_{i_p+2} - \overline{G}_{i_p})$ and such that $\overline{B(2)} \subseteq \varphi(U)$. Define

$$\psi_p = \begin{cases} b \circ \varphi & \text{on } U \\ 0 & \text{on } M - U, \end{cases}$$

where b is the bump function defined just before Proposition 3.23. Then, ψ_p is a smooth function on M which has value 1 on some open subset, W_p, containing p and has compact support lying in $U \subseteq U_{\alpha_p} \cap (G_{i_p+2} - \overline{G}_{i_p})$. For each $i \geq 1$, choose a finite set of points, $p \in M$, whose corresponding opens, W_p, cover $\overline{G}_i - G_{i-1}$. Order the corresponding ψ_p functions in a sequence, ψ_j, $j = 1, 2, \ldots$. The supports of the ψ_j form a locally finite family of subsets of M. Thus, the function

$$\psi = \sum_{j=1}^{\infty} \psi_j$$

is well-defined on M and smooth. Moreover, $\psi(p) > 0$ for each $p \in M$. For each $i \geq 1$, set

$$f_i = \frac{\psi_i}{\psi}.$$

Then, the family, $\{f_i\}_{i \geq 1}$, is a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ and $\mathrm{supp}\, f_i$ is compact for all $i \geq 1$.

Now, when we don't require compact support, if we let f_α be identically zero if no f_i has support in U_α and otherwise let f_α be the sum of the f_i with support in U_α, then we obtain a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in I}$ with at most countably many of the f_α not identically zero. We must have $\mathrm{supp}\, f_\alpha \subseteq U_\alpha$ because for any locally finite family of closed sets, $\{F_\beta\}_{\beta \in J}$, we have $\overline{\bigcup_{\beta \in J} F_\beta} = \bigcup_{\beta \in J} F_\beta$. $\square$

We close this section by stating a famous theorem of Whitney whose proof uses partitions of unity.

Theorem 3.27 *(Whitney, 1935) Any smooth manifold (Hausdorff and second-countable), M, of dimension n is diffeomorphic to a closed submanifold of $\mathbb{R}^{2n+1}$.*

For a proof, see Hirsch [76], Chapter 2, Section 2, Theorem 2.14.

3.7 Manifolds With Boundary

Up to now, we have defined manifolds locally diffeomorphic to an open subset of $\mathbb{R}^m$. This excludes many natural spaces such as a closed disk, whose boundary is a circle, a closed ball, $\overline{B(1)}$, whose boundary is the sphere, S^{m-1}, a compact cylinder, $S^1 \times [0, 1]$, whose boundary consist of two circles, a Möbius strip, etc. These spaces fail to be manifolds because they have a boundary, that is, neighborhoods of points on their boundaries are not diffeomorphic to open sets in $\mathbb{R}^m$. Perhaps the simplest example is the (closed) upper half space,

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

Under the natural emdedding $\mathbb{R}^{m-1} \cong \mathbb{R}^{m-1} \times \{0\} \hookrightarrow \mathbb{R}^m$, the subset $\partial\mathbb{H}^m$ of $\mathbb{H}^m$ defined by

$$\partial\mathbb{H}^m = \{x \in \mathbb{H}^m \mid x_m = 0\}$$

is isomorphic to $\mathbb{R}^{m-1}$ and is called the *boundary of* $\mathbb{H}^m$. We also define the *interior* of $\mathbb{H}^m$ as

$$\mathrm{Int}(\mathbb{H}^m) = \mathbb{H}^m - \partial\mathbb{H}^m.$$

Now, if U and V are open subsets of $\mathbb{H}^m$, where $\mathbb{H}^m \subseteq \mathbb{R}^m$ has the subset topology, and if $f\colon U \to V$ is a continuous function, we need to explain what we mean by f being smooth. We say that $f\colon U \to V$, as above, is *smooth* if it has an extension, $\widetilde{f}\colon \widetilde{U} \to \widetilde{V}$, where $\widetilde{U}$ and $\widetilde{V}$ are open subsets of $\mathbb{R}^m$ with $U \subseteq \widetilde{U}$ and $V \subseteq \widetilde{V}$ and with $\widetilde{f}$ a smooth function. We say that f is a (smooth) *diffeomorphism* iff f^{-1} exists and if both f and f^{-1} are smooth, as just defined.

To define a *manifold with boundary*, we replace everywhere $\mathbb{R}$ by $\mathbb{H}$ in Definition 3.1 and Definition 3.2. So, for instance, given a topological space, M, a *chart* is now pair, (U, φ), where U is an open subset of M and $\varphi\colon U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{H}^{n_\varphi}$ (for some $n_\varphi \geq 1$), etc. Thus, we obtain

Definition 3.26 Given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k-manifold of dimension n with boundary* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M (where the charts are now defined in terms of open subsets of $\mathbb{H}^n$). Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on* $\mathbb{H}^n$. When $k = \infty$, we say that M is a *smooth manifold with boundary*.

It remains to define what is the boundary of a manifold with boundary! By definition, the *boundary*, ∂M, of a manifold (with boundary), M, is the set of all points, $p \in M$, such that there is some chart, $(U_\alpha, \varphi_\alpha)$, with $p \in U_\alpha$ and $\varphi_\alpha(p) \in \partial\mathbb{H}^n$. We also let $\mathrm{Int}(M) = M - \partial M$ and call it the *interior* of M.

Do not confuse the boundary ∂M and the interior $\mathrm{Int}(M)$ of a manifold with boundary embedded in $\mathbb{R}^N$ with the topological notions of boundary and interior of M as a topological space. In general, they are different.

Note that manifolds as defined earlier (In Definition 3.3) are also manifolds with boundary: their boundary is just empty. We shall still reserve the word “manifold” for these, but for emphasis, we will sometimes call them “boundaryless”.

The definition of tangent spaces, tangent maps, etc., are easily extended to manifolds with boundary. The reader should note that if M is a manifold with boundary of dimension n, the tangent space, T_pM, is defined for all $p \in M$ and has dimension n, *even* for boundary points, $p \in \partial M$. The only notion that requires more care is that of a submanifold. For more on this, see Hirsch [76], Chapter 1, Section 4. One should also beware that the product of two manifolds with boundary is generally **not** a manifold with boundary (consider the product $[0,1] \times [0,1]$ of two line segments). There is a generalization of the notion of a manifold with boundary called *manifold with corners* and such manifolds are closed under products (see Hirsch [76], Chapter 1, Section 4, Exercise 12).

If M is a manifold with boundary, we see that $\mathrm{Int}(M)$ is a manifold without boundary. What about ∂M? Interestingly, the boundary, ∂M, of a manifold with boundary, M, of dimension n, is a manifold of dimension $n-1$. For this, we need the following Proposition:

Proposition 3.28 *If M is a manifold with boundary of dimension n, for any $p \in \partial M$ on the boundary on M, for any chart, (U, φ), with $p \in M$, we have $\varphi(p) \in \partial \mathbb{H}^n$.*

Proof. Since $p \in \partial M$, by definition, there is some chart, (V, ψ), with $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Let (U, φ) be any other chart, with $p \in M$ and assume that $q = \varphi(p) \in \mathrm{Int}(\mathbb{H}^n)$. The transition map, $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$, is a diffeomorphism and $q = \varphi(p) \in \mathrm{Int}(\mathbb{H}^n)$. By the inverse function theorem, there is some open, $W \subseteq \varphi(U \cap V) \cap \mathrm{Int}(\mathbb{H}^n) \subseteq \mathbb{R}^n$, with $q \in W$, so that $\psi \circ \varphi^{-1}$ maps W homeomorphically onto some subset, Ω, open in $\mathrm{Int}(\mathbb{H}^n)$, with $\psi(p) \in \Omega$, contradicting the hypothesis, $\psi(p) \in \partial \mathbb{H}^n$. □

Using Proposition 3.28, we immediately derive the fact that ∂M is a manifold of dimension $n-1$. We obtain charts on ∂M by considering the charts $(U \cap \partial M, L \circ \varphi)$, where (U, φ) is a chart on M such that $U \cap \partial M = \varphi^{-1}(\partial \mathbb{H}^n) \neq \emptyset$ and $L \colon \partial \mathbb{H}^n \to \mathbb{R}^{n-1}$ is the natural isomorphism (see see Hirsch [76], Chapter 1, Section 4).

3.8 Orientation of Manifolds

Although the notion of orientation of a manifold is quite intuitive it is technically rather subtle. We restrict our discussion to smooth manifolds (although the notion of orientation can also be defined for topological manifolds but more work is involved).

Intuitively, a manifold, M, is orientable if it is possible to give a consistent orientation to its tangent space, T_pM, at every point, $p \in M$. So, if we go around a closed curve starting at $p \in M$, when we come back to p, the orientation of T_pM should be the same as when we started. For exampe, if we travel on a Möbius strip (a manifold with boundary) dragging a coin with us, we will come back to our point of departure with the coin flipped. Try it!

To be rigorous, we have to say what it means to orient T_pM (a vector space) and what consistency of orientation means. We begin by quickly reviewing the notion of orientation of a vector space. Let E be a vector space of dimension n. If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of E, a basic and crucial fact of linear algebra says that there is a unique linear map, g, mapping each u_i to the corresponding v_i (i.e., $g(u_i) = v_i$, $i = 1, \ldots, n$). Then, look at the determinant, $\det(g)$, of this map. We know that $\det(g) = \det(P)$, where P is the matrix whose j-th columns consist of the coordinates of v_j over the basis $u_1, \ldots, u_n$. Either $\det(g)$ is negative or it is positive. Thus, we define an equivalence relation on bases by saying that two bases have the *same orientation* iff the determinant of the linear map sending the first basis to the second has positive determinant. An *orientation* of E is the choice of one of the two equivalence classes, which amounts to picking some basis as an orientation frame.

The above definition is perfectly fine but it turns out that it is more convenient, in the long term, to use a definition of orientation in terms of alternate multi-linear maps (in particular, to define the notion of integration on a manifold). Recall that a function, $h\colon E^k \to \mathbb{R}$, is *alternate multi-linear* (or *alternate k-linear*) iff it is linear in each of its arguments (holding the others fixed) and if

$$h(\ldots, x, \ldots, x, \ldots) = 0,$$

that is, h vanishes whenever two of its arguments are identical. Using multi-linearity, we immediately deduce that h vanishes for all k-tuples of arguments, $u_1, \ldots, u_k$, that are linearly dependent and that h is *skew-symmetric*, i.e.,

$$h(\ldots, y, \ldots, x, \ldots) = -h(\ldots, x, \ldots, y, \ldots).$$

In particular, for $k = n$, it is easy to see that if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases, then

$$h(v_1, \ldots, v_n) = \det(g)h(u_1, \ldots, u_n),$$

where g is the unique linear map sending each u_i to v_i. This shows that any alternating n-linear function is a multiple of the determinant function and that the space of alternating n-linear maps is a one-dimensional vector space that we will denote $\bigwedge^n E^*$.[1] We also call an alternating n-linear map an *n-form*. But then, observe that two bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ have the same orientation iff

$$\omega(u_1, \ldots, u_n) \quad \text{and} \quad \omega(v_1, \ldots, v_n) \quad \text{have the same sign for all } \omega \in \textstyle\bigwedge^n E^* - \{0\}$$

(where 0 denotes the zero n-form). As $\bigwedge^n E^*$ is one-dimensional, picking an orientation of E is equivalent to picking a generator (a one-element basis), ω, of $\bigwedge^n E^*$, and to say that $u_1, \ldots, u_n$ has positive orientation iff $\omega(u_1, \ldots, u_n) > 0$.

Given an orientation (say, given by $\omega \in \bigwedge^n E^*$) of E, a linear map, $f\colon E \to E$, is *orientation preserving* iff $\omega(f(u_1), \ldots, f(u_n)) > 0$ whenever $\omega(u_1, \ldots, u_n) > 0$ (or equivalently, iff $\det(f) > 0$).

Now, to define the orientation of an n-dimensional manifold, M, we use charts. Given any $p \in M$, for any chart, (U, φ), at p, the tangent map, $d\varphi^{-1}_{\varphi(p)}\colon \mathbb{R}^n \to T_pM$ makes sense. If $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$, as it gives an orientation to $\mathbb{R}^n$, we can orient T_pM by giving it the orientation induced by the basis $d\varphi^{-1}_{\varphi(p)}(e_1), \ldots, d\varphi^{-1}_{\varphi(p)}(e_n)$. Then, the consistency of orientations of the T_pM's is given by the overlapping of charts. We require that the Jacobian determinants of all $\varphi_j \circ \varphi_i^{-1}$ have the same sign, whenever (U_i, φ_i) and (U_j, φ_j) are any two overlapping charts. Thus, we are led to the definition below. All definitions and results stated in the rest of this section apply to manifolds with or without boundary.

[1] We are using the wedge product notation of exterior calculus even though we have not defined alternating tensors and the wedge product yet. This is standard notation and we hope that the reader will not be confused. In fact, in finite dimension, the space of alternating n-linear maps and $\bigwedge^n E^*$ are isomorphic. A thorough treatment of tensor algebra, including exterior algebra, and of differential forms, will be given in Chapters 19 and 8.

Definition 3.27 Given a smooth manifold, M, of dimension n, an *orientation atlas* of M is any atlas so that the transition maps, $\varphi_i^j = \varphi_j \circ \varphi_i^{-1}$, (from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$) all have a positive Jacobian determinant for every point in $\varphi_i(U_i \cap U_j)$. A manifold is *orientable* iff its has some orientation atlas.

Definition 3.27 can be hard to check in practice and there is an equivalent criterion is terms of n-forms which is often more convenient. The idea is that a manifold of dimension n is orientable iff there is a map, $p \mapsto \omega_p$, assigning to every point, $p \in M$, a nonzero n-form, $\omega_p \in \bigwedge^n T_p^* M$, so that this map is smooth. In order to explain rigorously what it means for such a map to be smooth, we can define the *exterior n-bundle*, $\bigwedge^n T^*M$ (also denoted $\bigwedge_n^* M$) in much the same way that we defined the bundles TM and T^*M. There is an obvious smooth projection map, $\pi\colon \bigwedge^n T^*M \to M$. Then, leaving the details of the fact that $\bigwedge^n T^*M$ can be made into a smooth manifold (of dimension n) as an exercise, a smooth map, $p \mapsto \omega_p$, is simply a smooth section of the bundle $\bigwedge^n T^*M$, i.e., a smooth map, $\omega\colon M \to \bigwedge^n T^*M$, so that $\pi \circ \omega = \mathrm{id}$.

Definition 3.28 If M is an n-dimensional manifold, a smooth section, $\omega \in \Gamma(M, \bigwedge^n T^*M)$, is called a (smooth) *n-form*. The set of n-forms, $\Gamma(M, \bigwedge^n T^*M)$, is also denoted $\mathcal{A}^n(M)$. An n-form, ω, is a *nowhere-vanishing n-form on M* or *volume form on M* iff ω_p is a nonzero form for every $p \in M$. This is equivalent to saying that $\omega_p(u_1, \ldots, u_n) \neq 0$, for all $p \in M$ and all bases, $u_1, \ldots, u_n$, of T_pM.

The determinant function, $(u_1, \ldots, u_n) \mapsto \det(u_1, \ldots, u_n)$, where the u_i are expressed over the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, is a volume form on $\mathbb{R}^n$. We will denote this volume form by ω_0. Another standard notation is $dx_1 \wedge \cdots \wedge dx_n$, but this notation may be very puzzling for readers not familiar with exterior algebra. Observe the justification for the term volume form: the quantity $\det(u_1, \ldots, u_n)$ is indeed the (signed) volume of the parallelepiped

$$\{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid 0 \leq \lambda_i \leq 1,\ 1 \leq i \leq n\}.$$

A volume form on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is obtained as follows:

$$\omega_p(u_1, \ldots u_n) = \det(p, u_1, \ldots u_n),$$

where $p \in S^n$ and $u_1, \ldots u_n \in T_pS^n$. As the u_i are orthogonal to p, this is indeed a volume form.

Observe that if f is a smooth function on M and ω is any n-form, then $f\omega$ is also an n-form.

Definition 3.29 Let $\varphi\colon M \to N$ be a smooth map of manifolds of the same dimension, n, and let $\omega \in \mathcal{A}^n(N)$ be an n-form on N. The *pull-back, $\varphi^*\omega$, of ω to M* is the n-form on M given by

$$\varphi^*\omega_p(u_1, \ldots, u_n) = \omega_{\varphi(p)}(d\varphi_p(u_1), \ldots, d\varphi_p(u_n)),$$

for all $p \in M$ and all $u_1, \ldots, u_n \in T_pM$.

One checks immediately that $\varphi^*\omega$ is indeed an n-form on M. More interesting is the following Proposition:

Proposition 3.29 *(a) If $\varphi\colon M \to N$ is a local diffeomorphism of manifolds, where* $\dim M = \dim N = n$*, and $\omega \in \mathcal{A}^n(N)$ is a volume form on N, then $\varphi^*\omega$ is a volume form on M. (b) Assume M has a volume form, ω. Then, for every n-form, $\eta \in \mathcal{A}^n(M)$, there is a unique smooth function, $f \in C^\infty(M)$, so that $\eta = f\omega$. If η is a volume form, then $f(p) \neq 0$ for all $p \in M$.*

Proof. (a) By definition,

$$\varphi^*\omega_p(u_1, \ldots, u_n) = \omega_{\varphi(p)}(d\varphi_p(u_1), \ldots, d\varphi_p(u_n)),$$

for all $p \in M$ and all $u_1, \ldots, u_n \in T_pM$. As φ is a local diffeomorphism, $d_p\varphi$ is a bijection for every p. Thus, if $u_1, \ldots, u_n$ is a basis, then so is $d\varphi_p(u_1), \ldots, d\varphi_p(u_n)$, and as ω is nonzero at every point for every basis, $\varphi^*\omega_p(u_1, \ldots, u_n) \neq 0$.

(b) Pick any $p \in M$ and let (U, φ) be any chart at p. As φ is a diffeomorphism, by (a), we see that ${\varphi^{-1}}^*\omega$ is a volume form on $\varphi(U)$. But then, it is easy to see that ${\varphi^{-1}}^*\eta = g{\varphi^{-1}}^*\omega$, for some unique smooth function, g, on $\varphi(U)$ and so, $\eta = f_U\omega$, for some unique smooth function, f_U, on U. For any two overlapping charts, (U_i, φ_i) and (U_j, φ_j), for every $p \in U_i \cap U_j$, for every basis $u_1, \ldots, u_n$ of T_pM, we have

$$\eta_p(u_1, \ldots, u_n) = f_i(p)\omega_p(u_1, \ldots, u_n) = f_j(p)\omega_p(u_1, \ldots, u_n),$$

and as $\omega_p(u_1, \ldots, u_n) \neq 0$, we deduce that f_i and f_j agree on $U_i \cap U_j$. But, then the f_i's patch on the overlaps of the cover, $\{U_i\}$, of M, and so, there is a smooth function, f, defined on the whole of M and such that $f \restriction U_i = f_i$. As the f_i's are unique, so is f. If η is a volume form, then η_p does not vanish for all $p \in M$ and since ω_p is also a volume form, ω_p does not vanish for all $p \in M$, so $f(p) \neq 0$ for all $p \in M$. $\square$

Remark: If φ and ψ are smooth maps of manifolds, it is easy to prove that

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$$

and that

$$\varphi^*(f\omega) = (f \circ \varphi)\varphi^*\omega,$$

where f is any smooth function on M and ω is any n-form.

The connection between Definition 3.27 and volume forms is given by the following important theorem whose proof contains a wonderful use of partitions of unity.

Theorem 3.30 *A smooth manifold (Hausdorff and second-countable) is orientable iff it possesses a volume form.*

Proof. First, assume that a volume form, ω, exists on M, and say $n = \dim M$. For any atlas, $\{(U_i, \varphi_i)\}_i$, of M, by Proposition 3.29, each n-form, $\varphi_i^{-1^*}\omega$, is a volume form on $\varphi_i(U_i) \subseteq \mathbb{R}^n$ and

$$\varphi_i^{-1^*}\omega = f_i\omega_0,$$

for some smooth function, f_i, never zero on $\varphi_i(U_i)$, where ω_0 is a volume form on $\mathbb{R}^n$. By composing φ_i with an orientation-reversing linear map if necessary, we may assume that for this new altlas, $f_i > 0$ on $\varphi_i(U_i)$. We claim that the family $(U_i, \varphi_i)_i$ is an orientation atlas. This is because, on any (nonempty) overlap, $U_i \cap U_j$, as $\omega = \varphi_j^*(f_j\omega_0)$ and $(\varphi_j \circ \varphi_i^{-1})^* = (\varphi_i^{-1})^* \circ \varphi_j^*$, we have

$$(\varphi_j \circ \varphi_i^{-1})^*(f_j\omega_0) = f_i\omega_0,$$

and by the definition of pull-backs, we see that for every $x \in \varphi_i(U_i \cap U_j)$, if we let $y = \varphi_j \circ \varphi_i^{-1}(x)$, then

$$\begin{aligned} f_i(x)(\omega_0)_x(e_1, \ldots, e_n) &= (\varphi_j \circ \varphi_i^{-1})_x^*(f_j\omega_0)(e_1, \ldots, e_n) \\ &= f_j(y)(\omega_0)_y d(\varphi_j \circ \varphi_i^{-1})_x(e_1), \ldots, d(\varphi_j \circ \varphi_i^{-1})_x(e_n)) \\ &= f_j(y)J((\varphi_j \circ \varphi_i^{-1})_x)(\omega_0)_y(e_1, \ldots, e_n), \end{aligned}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$ and $J((\varphi_j \circ \varphi_i^{-1})_x)$ is the Jacobian determinant of $\varphi_j \circ \varphi_i^{-1}$ at x. As both $f_j(y) > 0$ and $f_i(x) > 0$, we have $J((\varphi_j \circ \varphi_i^{-1})_x) > 0$, as desired.

Conversely, assume that $J((\varphi_j \circ \varphi_i^{-1})_x) > 0$, for all $x \in \varphi_i(U_i \cap U_j)$, whenever $U_i \cap U_j \neq \emptyset$. We need to make a volume form on M. For each U_i, let

$$\omega_i = \varphi_i^*\omega_0,$$

where ω_0 is a volume form on $\mathbb{R}^n$. As φ_i is a diffeomorphism, by Proposition 3.29, we see that ω_i is a volume form on U_i. Then, if we apply Theorem 3.26, we can find a partition of unity, $\{f_i\}$, subordinate to the cover $\{U_i\}$, with the same index set. Let,

$$\omega = \sum_i f_i\omega_i.$$

We claim that ω is a volume form on M.

It is clear that ω is an n-form on M. Now, since every $p \in M$ belongs to some U_i, check that on $\varphi_i(U_i)$, we have

$$\varphi_i^{-1^*}\omega = \sum_{j \in \text{finite set}} \varphi_i^{-1^*}(f_j\omega_j) = \left(\sum_j (f_j \circ \varphi_i^{-1})J(\varphi_j \circ \varphi_i^{-1})\right)\omega_0$$

and this sum is strictly positive because the Jacobian determinants are positive and as $\sum_j f_j = 1$ and $f_j \geq 0$, some term must be strictly positive. Therefore, $\varphi_i^{-1^*}\omega$ is a volume form on $\varphi_i(U_i)$ and so, $\varphi_i^* \varphi_i^{-1*} \omega = \omega$ is a volume form on U_i. As this holds for all U_i, we conclude that ω is a volume form on M. □

Since we showed that there is a volume form on the sphere, S^n, by Theorem 3.30, the sphere S^n is orientable. It can be shown that the projective spaces, $\mathbb{RP}^n$, are non-orientable iff n is even an thus, orientable iff n is odd. In particular, $\mathbb{RP}^2$ is not orientable. Also, even though M may not be orientable, its tangent bundle, $T(M)$, is always orientable! (Prove it). It is also easy to show that if $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth submersion, then $M = f^{-1}(0)$ is a smooth orientable manifold. Another nice fact is that every Lie group is orientable.

By Proposition 3.29 (b), given any two volume forms, ω_1 and ω_2 on a manifold, M, there is a function, $f\colon M \to \mathbb{R}$, never 0 on M such that $\omega_2 = f\omega_1$. This fact suggests the following definition:

Definition 3.30 Given an orientable manifold, M, two volume forms, ω_1 and ω_2, on M are *equivalent* iff $\omega_2 = f\omega_1$ for some smooth function, $f\colon M \to \mathbb{R}$, such that $f(p) > 0$ for all $p \in M$. An *orientation of* M is the choice of some equivalence class of volume forms on M and an *oriented manifold* is a manifold together with a choice of orientation. If M is a manifold oriented by the volume form, ω, for every $p \in M$, a basis, $(b_1, \ldots, b_n)$ of T_pM is *posively oriented* iff $\omega_p(b_1, \ldots, b_n) > 0$, else it is *negatively oriented* (where $n = \dim(M)$).

If M is an orientable manifold, for any two volume forms ω_1 and ω_2 on M, as $\omega_2 = f\omega_1$ for some function, f, on M which is never zero, f has a constant sign on every connected component of M. Consequently, a connected orientable manifold has two orientations.

We will also need the notion of orientation-preserving diffeomorphism.

Definition 3.31 Let $\varphi\colon M \to N$ be a diffeomorphism of oriented manifolds, M and N, of dimension n and say the orientation on M is given by the volume form ω_1 while the orientation on N is given by the volume form ω_2. We say that φ is *orientation preserving* iff $\varphi^*\omega_2$ determines the same orientation of M as ω_1.

Using Definition 3.31 we can define the notion of a positive atlas.

Definition 3.32 If M is a manifold oriented by the volume form, ω, an atlas for M is *positive* iff for every chart, (U, φ), the diffeomorphism, $\varphi\colon U \to \varphi(U)$, is orientation preserving, where U has the orientation induced by M and $\varphi(U) \subseteq \mathbb{R}^n$ has the orientation induced by the standard orientation on $\mathbb{R}^n$ (with $\dim(M) = n$).

The proof of Theorem 3.30 shows

Proposition 3.31 *If a manifold, M, has an orientation altas, then there is a uniquely determined orientation on M such that this atlas is positive.*

3.9 Covering Maps and Universal Covering Manifolds

Covering maps are an important technical tool in algebraic topology and more generally in geometry. This brief section only gives some basic definitions and states a few major facts. We apologize for his sketchy nature. Appendix A of O'Neill [117] gives a review of definitions and main results about covering manifolds. Expositions including full details can be found in Hatcher [71], Greenberg [65], Munkres [113] and Massey [103, 104] (the most extensive).

We begin with covering maps.

Definition 3.33 A map, $\pi\colon M \to N$, between two smooth manifolds is a *covering map* (or *cover*) iff

(1) The map π is smooth and surjective.

(2) For any $q \in N$, there is some open subset, $V \subseteq N$, so that $q \in V$ and

$$\pi^{-1}(V) = \bigcup_{i \in I} U_i,$$

where the U_i are pairwise disjoint open subsets, $U_i \subseteq M$, and $\pi\colon U_i \to V$ is a diffeomorphism for every $i \in I$. We say that V is *evenly covered*.

The manifold, M, is called a *covering manifold* of N.

A *homomorphism* of coverings, $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$, is a smooth map, $\varphi\colon M_1 \to M_2$, so that

$$\pi_1 = \pi_2 \circ \varphi,$$

that is, the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi} & M_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & N & \end{array}.$$

We say that the coverings $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are *equivalent* iff there is a homomorphism, $\varphi\colon M_1 \to M_2$, between the two coverings and φ is a diffeomorphism.

As usual, the inverse image, $\pi^{-1}(q)$, of any element $q \in N$ is called the *fibre over* q, the space N is called the *base* and M is called the *covering space*. As π is a covering map, each fibre is a discrete space. Note that a homomorphism maps each fibre $\pi_1^{-1}(q)$ in M_1 to the fibre $\pi_2^{-1}(\varphi(q))$ in M_2, for every $q \in M_1$.

Proposition 3.32 *Let $\pi\colon M \to N$ be a covering map. If N is connected, then all fibres, $\pi^{-1}(q)$, have the same cardinality for all $q \in N$. Furthermore, if $\pi^{-1}(q)$ is not finite then it is countably infinite.*

Proof. Pick any point, $p \in N$. We claim that the set

$$S = \{q \in N \mid |\pi^{-1}(q)| = |\pi^{-1}(p)|\}$$

is open and closed.

If $q \in S$, then there is some open subset, V, with $q \in V$, so that $\pi^{-1}(V)$ is evenly covered by some family, $\{U_i\}_{i \in I}$, of disjoint open subsets, U_i, each diffeomorphic to V under π. Then, every $s \in V$ must have a unique preimage in each U_i, so

$$|I| = |\pi^{-1}(s)|, \qquad \text{for all } s \in V.$$

However, as $q \in S$, $|\pi^{-1}(q)| = |\pi^{-1}(p)|$, so

$$|I| = |\pi^{-1}(p)| = |\pi^{-1}(s)|, \qquad \text{for all } s \in V,$$

and thus, $V \subseteq S$. Therefore, S is open. Similary the complement of S is open. As N is connected, $S = N$.

Since M is a manifold, it is second-countable, that is every open subset can be written as some countable union of open subsets. But then, every family, $\{U_i\}_{i \in I}$, of pairwise disjoint open subsets forming an even cover must be countable and since $|I|$ is the common cardinality of all the fibres, every fibre is countable. □

When the common cardinality of fibres is finite it is called the *multiplicity* of the covering (or the number of *sheets*).

For any integer, $n > 0$, the map, $z \mapsto z^n$, from the unit circle $S^1 = \mathbf{U}(1)$ to itself is a covering with n sheets. The map,

$$t\colon \ \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

is a covering, $\mathbb{R} \to S^1$, with infinitely many sheets.

It is also useful to note that a covering map, $\pi\colon M \to N$, is a local diffeomorphism (which means that $d\pi_p\colon T_pM \to T_{\pi(p)}N$ is a bijective linear map for every $p \in M$). Indeed, given any $p \in M$, if $q = \pi(p)$, then there is some open subset, $V \subseteq N$, containing q so that V is evenly covered by a family of disjoint open subsets, $\{U_i\}_{i \in I}$, with each $U_i \subseteq M$ diffeomorphic to V under π. As $p \in U_i$ for some i, we have a diffeomorphism, $\pi \restriction U_i\colon U_i \longrightarrow V$, as required.

The crucial property of covering manifolds is that curves in N can be lifted to M, in a unique way. For any map, $\varphi\colon P \to N$, a *lift of* φ *through* π is a map, $\widetilde{\varphi}\colon P \to M$, so that

$$\varphi = \pi \circ \widetilde{\varphi},$$

as in the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{\varphi}} & M \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \pi} \\ & & N \end{array}$$

We state without proof the following results:

Proposition 3.33 *If $\pi\colon M \to N$ is a covering map, then for every smooth curve, $\alpha\colon I \to N$, in N (with $0 \in I$) and for any point, $q \in M$, such that $\pi(q) = \alpha(0)$, there is a unique smooth curve, $\widetilde{\alpha}\colon I \to M$, lifting α through π such that $\widetilde{\alpha}(0) = q$.*

Proposition 3.34 *Let $\pi\colon M \to N$ be a covering map and let $\varphi\colon P \to N$ be a smooth map. For any $p_0 \in P$ and any $q_0 \in M$ with $\pi(q_0) = \varphi(p_0)$, the following properties hold:*

(1) If P is connected then there is at most one lift, $\widetilde{\varphi}\colon P \to M$, of φ through π such that $\widetilde{\varphi}(p_0) = q_0$.

(2) If P is simply connected, such a lift exists.

Theorem 3.35 *Every connected manifold, M, possesses a simply connected covering map, $\pi\colon \widetilde{M} \to M$, that is, with $\widetilde{M}$ simply connected. Any two simply connected coverings of N are equivalent.*

In view of Theorem 3.35, it is legitimate to speak of *the* simply connected cover, $\widetilde{M}$, of M, also called *universal covering* (or *cover*) of M.

Given any point, $p \in M$, let $\pi_1(M, p)$ denote the fundamental group of M with basepoint p (see any of the references listed above, in particular, Massey [103, 104]). If $\varphi\colon M \to N$ is a smooth map, for any $p \in M$, if we write $q = \varphi(p)$, then we have an induced group homomorphism

$$\varphi_*\colon \pi_1(M, p) \to \pi_1(N, q).$$

Proposition 3.36 *If $\pi\colon M \to N$ is a covering map, for every $p \in M$, if $q = \pi(p)$, then the induced homomorphism, $\pi_*\colon \pi_1(M, p) \to \pi_1(N, q)$, is injective.*

Basic Assumption: For any covering, $\pi\colon M \to N$, if N is connected then we also assume that M is connected.

Using Proposition 3.36, we get

Proposition 3.37 *If $\pi\colon M \to N$ is a covering map and N is simply connected, then π is a diffeomorphism (recall that M is connected); thus, M is diffeomorphic to the universal cover, $\widetilde{N}$, of N.*

Proof. Pick any $p \in M$ and let $q = \varphi(p)$. As N is simply connected, $\pi_1(N, q) = (0)$. By Proposition 3.36, since $\pi_*\colon \pi_1(M, p) \to \pi_1(N, q)$ is injective, $\pi_1(M, p) = (0)$ so M is simply connected (by hypothesis, M is connected). But then, by Theorem 3.35, M and N are diffeomorphic. $\square$

The following proposition shows that the universal covering of a space covers every other covering of that space. This justifies the terminology "universal covering".

Proposition 3.38 *Say $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are two coverings of N, with N connected. Every homomorphism, $\varphi\colon M_1 \to M_2$, between these two coverings is a covering map. As a consequence, if $\pi\colon \widetilde{N} \to N$ is a universal covering of N, then for every covering, $\pi'\colon M \to N$, of N, there is a covering, $\varphi\colon \widetilde{N} \to M$, of M.*

The notion of deck-transformation group of a covering is also useful because it yields a way to compute the fundamental group of the base space.

Definition 3.34 If $\pi\colon M \to N$ is a covering map, a *deck-transformation* is any diffeomorphism, $\varphi\colon M \to M$, such that $\pi = \pi \circ \varphi$, that is, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & M \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & N & \end{array}$$

Note that deck-transformations are just automorphisms of the covering map. The commutative diagram of Definition 3.34 means that a deck transformation permutes every fibre. It is immediately verified that the set of deck transformations of a covering map is a group denoted Γ_π (or simply, Γ), called the *deck-transformation group* of the covering.

Observe that any deck transformation, φ, is a lift of π through π. Consequently, if M is connected, by Proposition 3.34 (1), every deck-transformation is determined by its value at a single point. So, the deck-transformations are determined by their action on each point of any fixed fibre, $\pi^{-1}(q)$, with $q \in N$. Since the fibre $\pi^{-1}(q)$ is countable, Γ is also countable, that is, a discrete Lie group. Moreover, if M is compact, as each fibre, $\pi^{-1}(q)$, is compact and discrete, it must be finite and so, the deck-transformation group is also finite.

The following proposition gives a useful method for determining the fundamental group of a manifold.

Proposition 3.39 *If $\pi\colon \widetilde{M} \to M$ is the universal covering of a connected manifold, M, then the deck-transformation group, $\widetilde{\Gamma}$, is isomorphic to the fundamental group, $\pi_1(M)$, of M.*

Remark: When $\pi\colon \widetilde{M} \to M$ is the universal covering of M, it can be shown that the group $\widetilde{\Gamma}$ acts simply and transitively on every fibre, $\pi^{-1}(q)$. This means that for any two elements, $x, y \in \pi^{-1}(q)$, there is a unique deck-transformation, $\varphi \in \widetilde{\Gamma}$ such that $\varphi(x) = y$. So, there is a bijection between $\pi_1(M) \cong \widetilde{\Gamma}$ and the fibre $\pi^{-1}(q)$.

Proposition 3.35 together with previous observations implies that if the universal cover of a connected (compact) manifold is compact, then M has a finite fundamental group. We will use this fact later, in particular, in the proof of Myers' Theorem.