Chapter 4

Construction of Manifolds From Gluing Data

4.1 Sets of Gluing Data for Manifolds

The definition of a manifold given in Chapter 3 assumes that the underlying set, M, is already known. However, there are situations where we only have some indirect information about the overlap of the domains, U_i, of the local charts defining our manifold, M, in terms of the transition functions,

$$\varphi_{ji}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

but where M itself is not known. For example, this situation happens when trying to construct a surface approximating a 3D-mesh. If we let $\Omega_{ij} = \varphi_i(U_i \cap U_j)$ and $\Omega_{ji} = \varphi_j(U_i \cap U_j)$, then φ_{ji} can be viewed as a "gluing map",

$$\varphi_{ji}\colon \Omega_{ij} \to \Omega_{ji},$$

between two open subets of Ω_i and Ω_j, respectively.

For technical reasons, it is desirable to assume that the images, $\Omega_i = \varphi_i(U_i)$ and $\Omega_j = \varphi_j(U_j)$, of distinct charts are disjoint but this can always be achieved for manifolds. Indeed, the map

$$\beta\colon (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)$$

is a smooth diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B(0, 1)$ with inverse given by

$$\beta^{-1}\colon (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right).$$

Since M has a countable basis, using compositions of β with suitable translations, we can make sure that the Ω_i's are mapped diffeomorphically to disjoint open subsets of $\mathbb{R}^n$.

Remarkably, manifolds can be constructed using the "gluing process" alluded to above from what is often called sets of "gluing data." In this chapter, we are going to describe this construction and prove its correctness in details, provided some mild assumptions on the gluing data. It turns out that this procedure for building manifolds can be made practical. Indeed, it is the basis of a class of new methods for approximating 3D meshes by smooth surfaces, see Siqueira, Xu and Gallier [138].

It turns out that care must be exercised to ensure that the space obtained by gluing the pieces Ω_{ij} and Ω_{ji} is Hausdorff. Some care must also be exercised in formulating the consistency conditions relating the φ_{ji}'s (the so-called "cocycle condition"). This is because the traditional condition (for example, in bundle theory) has to do with triple overlaps of the $U_i = \theta_i(\Omega_i)$ on the manifold, M, (see Chapter 7, especially Theorem 7.4) but in our situation, we do not have M nor the parametrization maps θ_i and the cocycle condition on the φ_{ji}'s has to be stated in terms of the Ω_i's and the Ω_{ji}'s.

Finding an easily testable necessary and sufficient criterion for the Hausdorff condition appears to be a very difficult problem. We propose a necessary and sufficient condition, but it is not easily testable in general. If M is a manifold, then observe that difficulties may arise when we want to separate two distinct point, $p, q \in M$, such that p and q neither belong to the same open, $\theta_i(\Omega_i)$, nor to two disjoint opens, $\theta_i(\Omega_i)$ and $\theta_j(\Omega_j)$, but instead, to the boundary points in $(\partial(\theta_i(\Omega_{ij})) \cap \theta_i(\Omega_i)) \cup (\partial(\theta_i(\Omega_{ji})) \cap \theta_j(\Omega_j))$. In this case, there are some disjoint open subsets, U_p and U_q, of M with $p \in U_p$ and $q \in U_q$, and we get two disjoint open subsets, $V_x = \theta_i^{-1}(U_p) \subseteq \Omega_i$ and $V_y = \theta_j^{-1}(U_q) \subseteq \Omega_j$, with $\theta_i(x) = p$, $\theta_j(y) = q$, and such that $x \in \partial(\Omega_{ij}) \cap \Omega_i$, $y \in \partial(\Omega_{ji}) \cap \Omega_j$, and no point in $V_y \cap \Omega_{ji}$ is the image of any point in $V_x \cap \Omega_{ij}$ by φ_{ji}. Since V_x and V_y are open, we may assume that they are open balls. This necessary condition turns out to be also sufficient.

With the above motivations in mind, here is the definition of sets of gluing data.

Definition 4.1 Let n be an integer with $n \geq 1$ and let k be either an integer with $k \geq 1$ or $k = \infty$. A *set of gluing data* is a triple, $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$, satisfying the following properties, where I is a (nonempty) countable set:

(1) For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{R}^n$ called a *parametrization domain*, for short, *p-domain*, and the Ω_i are pairwise disjoint (*i.e.*, $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ij} \neq \emptyset$ iff $\Omega_{ji} \neq \emptyset$. Each nonempty Ω_{ij} (with $i \neq j$) is called a *gluing domain*.

(3) If we let

$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then $\varphi_{ji} \colon \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i, j) \in K$ called a *transition function* (or *gluing function*) and the following condition holds:

(c) For all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x), \qquad \text{for all} \quad x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}).$$

Condition (c) is called the *cocycle* condition.

(4) For every pair $(i, j) \in K$, with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and every $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji}.

Remarks.

(1) In practical applications, the index set, I, is of course finite and the open subsets, Ω_i, may have special properties (for example, connected; open simplicies, *etc.*).

(2) We are only interested in the Ω_{ij}'s that are nonempty but empty Ω_{ij}'s do arise in proofs and constructions and this is why our definition allows them.

(3) Observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, as Ω_i and Ω_j are disjoint, so are Ω_{ij} and Ω_{ij}.

(4) The cocycle condition (c) may seem overly complicated but it is actually needed to guarantee the transitivity of the relation, $\sim$, defined in the proof of Proposition 4.1. Flawed versions of condition (c) appear in the literature, see the discussion after the proof of Proposition 4.1. The problem is that $\varphi_{kj} \circ \varphi_{ji}$ is a partial function whose domain, $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, is not necessarily related to the domain, Ω_{ik}, of φ_{ki}. To ensure the transitivity of $\sim$, we must assert that whenever the composition $\varphi_{kj} \circ \varphi_{ji}$ has a nonempty domain, this domain is contained in the domain of φ_{ki} and that $\varphi_{kj} \circ \varphi_{ji}$ and φ_{ki} agree. Since the φ_{ji} are bijections, condition (c) implies the following conditions:

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$.

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$.

To get (a), set $i = j = k$. Then, (b) follows from (a) and (c) by setting $k = i$.

(5) If M is a C^k manifold (including $k = \infty$), then using the notation of our introduction, it is easy to check that the open sets Ω_i, Ω_{ij} and the gluing functions, φ_{ji}, satisfy the conditions of Definition 4.1 (provided that we fix the charts so that the images of distinct charts are disjoint). Proposition 4.1 will show that a manifold can be reconstructed from a set of guing data.

The idea of defining gluing data for manifolds is not new. André Weil introduced this idea to define abstract algebraic varieties by gluing irreducible affine sets in his book [146] published in 1946. The same idea is well-known in bundle theory and can be found in standard texts such as Steenrod [139], Bott and Tu [19], Morita [112] and Wells [148] (the construction of a fibre bundle from a cocycle is given in Chapter 7, see Theorem 7.4).

The beauty of the idea is that it allows the reconstruction of a manifold, M, without having prior knowledge of the topology of this manifold (that is, without having explicitly the underlying topological space M) by gluing open subets of $\mathbb{R}^n$ (the Ω_i's) according to prescribed gluing instructions (namely, glue Ω_i and Ω_j by identifying Ω_{ij} and Ω_{ji} using φ_{ji}). This method of specifying a manifold separates clearly the local structure of the manifold (given by the Ω_i's) from its global structure which is specified by the gluing functions. Furthermore, this method ensures that the resulting manifold is C^k (even for $k = \infty$) with no extra effort since the gluing functions φ_{ji} are assumed to be C^k.

Grimm and Hughes [67, 68] appear to be the first to have realized the power of this latter property for practical applications and we wish to emphasize that this is a very significant discovery. However, Grimm [67] uses a condition much stronger than our condition (4) to ensure that the resulting space is Hausdorff. Their cocycle condition is also too weak to ensure transitivity of the relation $\sim$. We will come back to these points after the proof of Proposition 4.1.

Working with overlaps of *open subsets* of the parameter domain makes it much easier to enforce smoothness conditions compared to the traditional approach with splines where the parameter domain is subdivided into *closed* regions and where enforcing smoothness along boundaries is much more difficult.

Let us show that a set of gluing data defines a C^k manifold in a natural way.

Proposition 4.1 *For every set of gluing data, $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$, there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the φ_{ji}'s.*

Proof. Define the binary relation, $\sim$, on the disjoint union, $\coprod_{i\in I}\Omega_i$, of the open sets, Ω_i, as follows: For all $x, y \in \coprod_{i\in I}\Omega_i$,

$$x \sim y \quad \text{iff} \quad (\exists (i,j) \in K)(x \in \Omega_{ij},\ y \in \Omega_{ji},\ y = \varphi_{ji}(x)).$$

Note that if $x \sim y$ and $x \neq y$, then $i \neq j$, as $\varphi_{ii} = \mathrm{id}$. But then, as $x \in \Omega_{ij} \subseteq \Omega_i$, $y \in \Omega_{ji} \subseteq \Omega_j$ and $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$, if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$.

We claim that $\sim$ is an equivalence relation. This follows easily from the cocycle condition but to be on the safe side, we provide the crucial step of the proof. Clearly, condition (a) ensures reflexivity and condition (b) ensures symmetry. The crucial step is to check transitivity. Assume that $x \sim y$ and $y \sim z$. Then, there are some i, j, k such that

(i) $x \in \Omega_{ij}$, $y \in \Omega_{ji} \cap \Omega_{jk}$, $z \in \Omega_{kj}$ and

(ii) $y = \varphi_{ji}(x)$ and $z = \varphi_{kj}(y)$.

Consequently, $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ and $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, so by (c), we get $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and thus, $\varphi_{ki}(x)$ is defined and by (c) again,

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) = z,$$

that is, $x \sim z$, as desired.

Since $\sim$ is an equivalence relation let

$$M_{\mathcal{G}} = \Big(\coprod_{i \in I} \Omega_i\Big) / \sim$$

be the quotient set and let $p\colon \coprod_{i \in I} \Omega_i \to M_{\mathcal{G}}$ be the quotient map, with $p(x) = [x]$, where $[x]$ denotes the equivalence class of x. Also, for every $i \in I$, let $\mathrm{in}_i\colon \Omega_i \to \coprod_{i \in I} \Omega_i$ be the natural injection and let

$$\tau_i = p \circ \mathrm{in}_i\colon \Omega_i \to M_{\mathcal{G}}.$$

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we conclude that every τ_i is injective.

We give $M_{\mathcal{G}}$ the coarsest topology that makes the bijections, $\tau_i\colon \Omega_i \to \tau_i(\Omega_i)$, into homeomorphisms. Then, if we let $U_i = \tau_i(\Omega_i)$ and $\varphi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, φ_i) are charts and this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. As there are countably many charts, $M_{\mathcal{G}}$ is second-countable. Therefore, for $M_{\mathcal{G}}$ to be a manifold it only remains to check that the topology is Hausdorff. For this, we use the following:

Claim. For all $(i, j) \in I \times I$, we have $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i, j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Proof of Claim. Assume that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ and let $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$. Observe that $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$ iff $z \sim x$ and $z \sim y$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Consequently, $x \sim y$, which implies that $(i, j) \in K$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$.

We have $[z] \in \tau_i(\Omega_{ij})$ iff $z \sim x$ for some $x \in \Omega_{ij}$. Then, either $i = j$ and $z = x$ or $i \neq j$ and $z \in \Omega_{ji}$, which shows that $[z] \in \tau_j(\Omega_{ji})$ and so,

$$\tau_i(\Omega_{ij}) \subseteq \tau_j(\Omega_{ji}).$$

Since the same argument applies by interchanging i and j, we have

$$\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}),$$

for all $(i, j) \in K$. Since $\Omega_{ij} \subseteq \Omega_i$, $\Omega_{ji} \subseteq \Omega_j$ and $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$ for all $(i, j) \in K$, we have

$$\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}) \subseteq \tau_i(\Omega_i) \cap \tau_j(\Omega_j),$$

for all $(i, j) \in K$.

For the reverse inclusion, if $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, then we know that there is some $x \in \Omega_{ij}$ and some $y \in \Omega_{ji}$ such that $z \sim x$ and $z \sim y$, so $[z] = [x] \in \tau_i(\Omega_{ij})$, $[z] = [y] \in \tau_j(\Omega_{ji})$ and we get

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \subseteq \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

This proves that if $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$, then $(i,j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Finally, assume $(i,j) \in K$. Then, for any $x \in \Omega_{ij} \subseteq \Omega_i$, we have $y = \varphi_{ji}(x) \in \Omega_{ji} \subseteq \Omega_j$ and $x \sim y$, so that $\tau_i(x) = \tau_j(y)$, which proves that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ and our claim is proved.

We now prove that the topology of $M_{\mathcal{G}}$ is Hausdorff. Pick $[x], [y] \in M_{\mathcal{G}}$ with $[x] \neq [y]$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Either $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \emptyset$, in which case, as τ_i and τ_j are homeomorphisms, $[x]$ and $[y]$ belong to the two disjoint open sets $\tau_i(\Omega_i)$ and $\tau_j(\Omega_j)$. If not, then by the Claim, $(i,j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

There are several cases to consider:

(a) If $i = j$, then x and y can be separated by disjoint opens, V_x and V_y, and as τ_i is a homeomorphism, $[x]$ and $[y]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_i(V_y)$.

(b) If $i \neq j$, $x \in \Omega_i - \overline{\Omega_{ij}}$ and $y \in \Omega_j - \overline{\Omega_{ji}}$, then $\tau_i(\Omega_i - \overline{\Omega_{ij}})$ and $\tau_j(\Omega_j - \overline{\Omega_{ji}})$ are disjoint opens subsets separating $[x]$ and $[y]$.

(c) If $i \neq j$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$, as $[x] \neq [y]$ and $y \sim \varphi_{ij}(y)$, then $x \neq \varphi_{ij}(y)$. We can separate x and $\varphi_{ij}(y)$ by disjoint open subsets, V_x and V_y and $[x]$ and $[y] = [\varphi_{ij}(y)]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_i(V_y)$.

(d) If $i \neq j$, $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then we use condition (4). This condition yields two disjoint open subsets V_x and V_y with $x \in V_x$ and $y \in V_y$ such that no point of $V_x \cap \Omega_{ij}$ is equivalent to any point of $V_y \cap \Omega_{ji}$, and so, $\tau_i(V_x)$ and $\tau_j(V_y)$ are disjoint open subsets separating $[x]$ and $[y]$.

Therefore, the topology of $M_{\mathcal{G}}$ is Hausdorff and $M_{\mathcal{G}}$ is indeed a manifold.

Finally, it is trivial to verify that the transition functions of $M_{\mathcal{G}}$ are the original gluing functions, φ_{ij}. $\square$

It should be noted that as nice as it is, Proposition 4.1 is a theoretical construction that yields an "abstract" manifold but does not yield any information as to the geometry of this manifold. Furthermore, the resulting manifold may not be orientable or compact, even if we start with a finite set of p-domains.

Here is an example showing that if condition (4) of Definition 4.1 is omitted then we may get non-Hausdorff spaces. Cindy Grimm uses a similar example in her dissertation [67], but her presentation is somewhat confusing because her Ω_1 and Ω_2 appear to be two disjoint copies of the real line in $\mathbb{R}^2$, but these are not open in $\mathbb{R}^2$!

Let $\Omega_1 = (-3, -1)$, $\Omega_2 = (1, 3)$, $\Omega_{12} = (-3, -2)$, $\Omega_{21} = (1, 2)$ and $\varphi_{21}(x) = x + 4$. The resulting space, M, is a curve looking like a "fork", and the problem is that the images of -2 and 2 in M, which are distinct points of M, cannot be separated. Indeed, the images of any two open intervals $(-2 - \epsilon, -2 + \epsilon)$ and $(2 - \eta, 2 + \eta)$ (for $\epsilon, \eta > 0$) always intersect since $(-2 - \min(\epsilon, \eta), -2)$ and $(2 - \min(\epsilon, \eta), 2)$ are identified. Clearly, condition (4) fails.

Cindy Grimm [67] uses a condition stronger than our condition (4) to ensure that the quotient, $M_{\mathcal{G}}$ is Hausdorff, namely, that for all $i, j \in K$ with $i \neq j$, the quotient $(\Omega_i \coprod \Omega_j)/\sim$ should be embeddable in $\mathbb{R}^n$. This is a rather strong condition that prevents obtaining a 2-sphere by gluing two open discs in $\mathbb{R}^2$ along an annulus (see Grimm [67], Appendix C).

Grimm [67, 68] uses the following cocycle condition:

(c') For all $x \in \Omega_{ij} \cap \Omega_{ik}$,

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x).$$

This condition is too weak to imply transitivity of the relation $\sim$, as shown by the following counter-example:

Let $\Omega_1 = (0, 3)$, $\Omega_2 = (4, 5)$, $\Omega_3 = (6, 9)$, $\Omega_{12} = (0, 1)$, $\Omega_{13} = (2, 3)$, $\Omega_{21} = \Omega_{23} = (4, 5)$, $\Omega_{32} = (8, 9)$, $\Omega_{31} = (6, 7)$, $\varphi_{21}(x) = x + 4$, $\varphi_{32}(x) = x + 4$ and $\varphi_{31}(x) = x + 4$.

Note that the pairwise gluings yield Hausdorff spaces. Obviously, $\varphi_{32} \circ \varphi_{21}(x) = x + 8$, for all $x \in \Omega_{12}$, but $\Omega_{12} \cap \Omega_{13} = \emptyset$. Thus, $0.5 \sim 4.5 \sim 8.5$, but $0.5 \not\sim 8.5$ since $\varphi_{31}(0.5)$ is undefined.

Here is another counter-example in which $\Omega_{12} \cap \Omega_{13} \neq \emptyset$, using a disconnected open, Ω_2.

Let $\Omega_1 = (0, 3)$, $\Omega_2 = (4, 5) \cup (6, 7)$, $\Omega_3 = (8, 11)$, $\Omega_{12} = (0, 1) \cup (2, 3)$, $\Omega_{13} = (2, 3)$, $\Omega_{21} = \Omega_{23} = (4, 5) \cup (6, 7)$, $\Omega_{32} = (8, 9) \cup (10, 11)$, $\Omega_{31} = (8, 9)$, $\varphi_{21}(x) = x + 4$, $\varphi_{32}(x) = x + 2$ on $(6, 7)$, $\varphi_{32}(x) = x + 6$ on $(4, 5)$, $\varphi_{31}(x) = x + 6$.

Note that the pairwise gluings yield Hausdorff spaces. Obviously, $\varphi_{32} \circ \varphi_{21}(x) = x + 6 = \varphi_{31}(x)$ for all $x \in \Omega_{12} \cap \Omega_{13} = (2, 3)$. Thus, $0.5 \sim 4.5 \sim 8.5$, but $0.5 \not\sim 8.5$ since $\varphi_{31}(0.5)$ is undefined.

It is possible to give a construction, in the case of a surface, which builds a compact manifold whose geometry is "close" to the geometry of a prescribed 3D-mesh (see Siqueira, Xu and Gallier [138]). Actually, we are not able to guarantee, in general, that the parametrization functions, θ_i, that we obtain are injective, but we are not aware of any algorithm that achieves this.

Given a set of gluing data, $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K})$, it is natural to consider the collection of manifolds, M, parametrized by maps, $\theta_i \colon \Omega_i \to M$, whose domains are the

Ω_i's and whose transitions functions are given by the φ_{ji}, that is, such that

$$\varphi_{ji} = \theta_j^{-1} \circ \theta_i.$$

We will say that such manifolds are *induced* by the set of gluing data, $\mathcal{G}$.

The proof of Proposition 4.1 shows that the parametrization maps, τ_i, satisfy the property: $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i, j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Furthermore, they also satisfy the consistency condition:

$$\tau_i = \tau_j \circ \varphi_{ji},$$

for all $(i, j) \in K$. If M is a manifold induced by the set of gluing data, $\mathcal{G}$, because the θ_i's are injective and $\varphi_{ji} = \theta_j^{-1} \circ \theta_i$, the two properties stated above for the τ_i's also hold for the θ_i's. We will see in Section 4.2 that the manifold, $M_{\mathcal{G}}$, is a "universal" manifold induced by $\mathcal{G}$ in the sense that every manifold induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by some C^k map.

Interestingly, it is possible to characterize when two manifolds induced by the same set of gluing data are isomorphic in terms of a condition on their transition functions.

Proposition 4.2 *Given any set of gluing data, $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$, for any two manifolds M and M' induced by $\mathcal{G}$ given by families of parametrizations $(\Omega_i, \theta_i)_{i\in I}$ and $(\Omega_i, \theta'_i)_{i\in I}$, respectively, if $f\colon M \to M'$ is a C^k isomorphism, then there are C^k bijections, $\rho_i\colon W_{ij} \to W'_{ij}$, for some open subsets $W_{ij}, W'_{ij} \subseteq \Omega_i$, such that*

$$\varphi'_{ji}(x) = \rho_j \circ \varphi_{ji} \circ \rho_i^{-1}(x), \qquad \textit{for all} \quad x \in W_{ij},$$

with $\varphi_{ji} = \theta_j^{-1} \circ \theta_i$ and $\varphi'_{ji} = \theta_j'^{-1} \circ \theta'_i$. Furthermore, $\rho_i = (\theta_i'^{-1} \circ f \circ \theta_i) \restriction W_{ij}$ and if $\theta_i'^{-1} \circ f \circ \theta_i$ is a bijection from Ω_i to itself and $\theta_i'^{-1} \circ f \circ \theta_i(\Omega_{ij}) = \Omega_{ij}$, for all i, j, then $W_{ij} = W'_{i,j} = \Omega_i$.

Proof. The composition $\theta_i'^{-1} \circ f \circ \theta_i$ is actually a partial function with domain

$$\mathrm{dom}(\theta_i'^{-1} \circ f \circ \theta_i) = \{x \in \Omega_i \mid \theta_i(x) \in f^{-1} \circ \theta'_i(\Omega_i)\}$$

and its "inverse" $\theta_i^{-1} \circ f^{-1} \circ \theta'_i$ is a partial function with domain

$$\mathrm{dom}(\theta_i^{-1} \circ f^{-1} \circ \theta'_i) = \{x \in \Omega_i \mid \theta'_i(x) \in f \circ \theta_i(\Omega_i)\}.$$

The composition $\theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta'_i$ is also a partial function and we let

$$W_{ij} = \Omega_{ij} \cap \mathrm{dom}(\theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta'_i), \qquad \rho_i = (\theta_i'^{-1} \circ f \circ \theta_i) \restriction W_{ij}$$

and $W'_{ij} = \rho_i(W_{ij})$. Observe that $\theta_j \circ \varphi_{ji} = \theta_j \circ \theta_j^{-1} \circ \theta_i = \theta_i$, that is,

$$\theta_i = \theta_j \circ \varphi_{ji}.$$

Using this, on W_{ij}, we get

$$\begin{aligned}
\rho_j \circ \varphi_{ji} \circ \rho_i^{-1} &= \theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ (\theta_i'^{-1} \circ f \circ \theta_i)^{-1} \\
&= \theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta'_i \\
&= \theta_j'^{-1} \circ f \circ \theta_i \circ \theta_i^{-1} \circ f^{-1} \circ \theta'_i \\
&= \theta_j'^{-1} \circ \theta'_i = \varphi'_{ji},
\end{aligned}$$

as claimed. The last part of the proposition is clear. □

Proposition 4.2 suggests defining a notion of equivalence on sets of gluing data which yields a converse of this proposition.

Definition 4.2 Two sets of gluing data, $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$ and $\mathcal{G}' = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi'_{ji})_{(i,j)\in K})$, over the same sets of Ω_i's and Ω_{ij}'s are *equivalent* iff there is a family of C^k bijections, $(\rho_i \colon \Omega_i \to \Omega_i)_{i\in I}$, such that $\rho_i(\Omega_{ij}) = \Omega_{ij}$ and

$$\varphi'_{ji}(x) = \rho_j \circ \varphi_{ji} \circ \rho_i^{-1}(x), \qquad \text{for all} \quad x \in \Omega_{ij},$$

for all i, j.

Here is the converse of Proposition 4.2. It is actually nicer than Proposition 4.2 because we can take $W_{ij} = W'_{ij} = \Omega_i$.

Proposition 4.3 *If two sets of gluing data* $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$ *and* $\mathcal{G}' = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi'_{ji})_{(i,j)\in K})$ *are equivalent, then there is a* C^k *isomorphism,* $f \colon M_{\mathcal{G}} \to M_{\mathcal{G}'}$*, between the manifolds induced by* $\mathcal{G}$ *and* $\mathcal{G}'$*. Furthermore,* $f \circ \tau_i = \tau'_i \circ \rho_i$*, for all* $i \in I$*.*

Proof. Let $f_i \colon \tau_i(\Omega_i) \to \tau'_i(\Omega_i)$ be the C^k bijection given by

$$f_i = \tau'_i \circ \rho_i \circ \tau_i^{-1},$$

where the $\rho_i \colon \Omega_i \to \Omega_i$'s are the maps giving the equivalence of $\mathcal{G}$ and $\mathcal{G}'$. If we prove that f_i and f_j agree on the overlap, $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$, then the f_i patch and yield a C^k isomorphism, $f \colon M_{\mathcal{G}} \to M_{\mathcal{G}'}$. The conditions of Proposition 4.3 imply that

$$\varphi'_{ji} \circ \rho_i = \rho_j \circ \varphi_{ji}$$

and we know that

$$\tau'_i = \tau'_j \circ \varphi'_{ji}.$$

Consequently, for every $[x] \in \tau_j(\Omega_{ji}) = \tau_i(\Omega_{ij})$, with $x \in \Omega_{ij}$, we have

$$\begin{aligned} f_j([x]) &= \tau_j' \circ \rho_j \circ \tau_j^{-1}([x]) \\ &= \tau_j' \circ \rho_j \circ \tau_j^{-1}([\varphi_{ji}(x)]) \\ &= \tau_j' \circ \rho_j \circ \varphi_{ji}(x) \\ &= \tau_j' \circ \varphi_{ji}' \circ \rho_i(x) \\ &= \tau_i' \circ \rho_i(x) \\ &= \tau_i' \circ \rho_i \circ \tau_i^{-1}([x]) \\ &= f_i([x]), \end{aligned}$$

which shows that f_i and f_j agree on $\tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, as claimed. $\square$

In the next section, we describe a class of spaces that can be defined by gluing data and parametrization functions, θ_i, that are not necessarily injective. Roughly speaking, the gluing data specify the topology and the parametrizations define the geometry of the space. Such spaces have more structure than spaces defined parametrically but they are not quite manifolds. Yet, they arise naturally in practice and they are the basis of efficient implementations of very good approximations of 3D meshes.

4.2 Parametric Pseudo-Manifolds

In practice, it is often desirable to specify some n-dimensional geometric shape as a subset of $\mathbb{R}^d$ (usually for $d = 3$) in terms of parametrizations which are functions, θ_i, from some subset of $\mathbb{R}^n$ into $\mathbb{R}^d$ (usually, $n = 2$). For "open" shapes, this is reasonably well understood but dealing with a "closed" shape is a lot more difficult because the parametrized pieces should overlap as smoothly as possible and this is hard to achieve. Furthermore, in practice, the parametrization functions, θ_i, may not be injective. Proposition 4.1 suggests various ways of defining such geometric shapes. For the lack of a better term, we will call these shapes, *parametric pseudo-manifolds*.

Definition 4.3 Let n, k, d be three integers with $d > n \geq 1$ and $k \geq 1$ or $k = \infty$. A *parametric C^k pseudo-manifold of dimension n in $\mathbb{R}^d$* is a pair, $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I})$, where $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K})$ is a set of gluing data for some finite set, I, and each θ_i is a C^k function, $\theta_i \colon \Omega_i \to \mathbb{R}^d$, called a *parametrization* such that the following property holds:

(C) For all $(i, j) \in K$, we have

$$\theta_i = \theta_j \circ \varphi_{ji}.$$

For short, we use terminology *parametric pseudo-manifold*. The subset, $M \subseteq \mathbb{R}^d$, given by

$$M = \bigcup_{i \in I} \theta_i(\Omega_i)$$

is called the *image* of the parametric pseudo-manifold, $\mathcal{M}$. When $n = 2$ and $d = 3$, we say that $\mathcal{M}$ is a *parametric pseudo-surface*.

Condition (C) obviously implies that

$$\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}),$$

for all $(i, j) \in K$. Consequently, θ_i and θ_j are consistent parametrizations of the overlap, $\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji})$. Thus, the shape, M, is covered by pieces, $U_i = \theta_i(\Omega_i)$, not necessarily open, with each U_i parametrized by θ_i and where the overlapping pieces, $U_i \cap U_j$, are parametrized consistently. The local structure of M is given by the θ_i's and the global structure is given by the gluing data. We recover a manifold if we require the θ_i to be bijective and to satisfy the following additional conditions:

(C') For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}).$$

(C") For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset.$$

Even if the θ_i's are not injective, properties (C') and (C") would be desirable since they guarantee that $\theta_i(\Omega_i - \Omega_{ij})$ and $\theta_j(\Omega_j - \Omega_{ji})$ are parametrized uniquely. Unfortunately, these properties are difficult to enforce. Observe that any manifold induced by $\mathcal{G}$ is the image of a parametric pseudo-manifold.

Although this is an abuse of language, it is more convenient to call M a parametric pseudo-manifold, or even a *pseudo-manifold*.

We can also show that the parametric pseudo-manifold, M, is the image in $\mathbb{R}^d$ of the abstract manifold, $M_{\mathcal{G}}$.

Proposition 4.4 *Let $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$ be parametric C^k pseudo-manifold of dimension n in $\mathbb{R}^d$, where $\mathcal{G} = ((\Omega_i)_{\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$ is a set of gluing data for some finite set, I. Then, the parametrization maps, θ_i, induce a surjective map, $\Theta\colon M_{\mathcal{G}} \to M$, from the abstract manifold, $M_{\mathcal{G}}$, specified by $\mathcal{G}$ to the image, $M \subseteq \mathbb{R}^d$, of the parametric pseudo-manifold, $\mathcal{M}$, and the following property holds: For every Ω_i,*

$$\theta_i = \Theta \circ \tau_i,$$

where the $\tau_i\colon \Omega_i \to M_{\mathcal{G}}$ are the parametrization maps of the manifold $M_{\mathcal{G}}$ (see Proposition 4.1). In particular, every manifold, M, induced by the gluing data $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by a map $\Theta\colon M_{\mathcal{G}} \to M$.

Proof. Recall that
$$M_{\mathcal{G}} = \Big(\coprod_{i\in I} \Omega_i\Big)/\sim,$$
where $\sim$ is the equivalence relation defined so that, for all $x, y \in \coprod_{i\in I} \Omega_i$,
$$x \sim y \quad \text{iff} \quad (\exists (i,j) \in K)(x \in \Omega_{ij},\ y \in \Omega_{ji},\ y = \varphi_{ji}(x)).$$
The proof of Proposition 4.1 also showed that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i,j) \in K$ and if so,
$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$
In particular,
$$\tau_i(\Omega_i - \Omega_{ij}) \cap \tau_j(\Omega_j - \Omega_{ji}) = \emptyset$$
for all $(i,j) \in I \times I$ ($\Omega_{ij} = \Omega_{ji} = \emptyset$ when $(i,j) \notin K$). These properties with the fact that the τ_i's are injections show that for all $(i,j) \notin K$, we can define $\Theta_i \colon \tau_i(\Omega_i) \to \mathbb{R}^d$ and $\Theta_j \colon \tau_i(\Omega_j) \to \mathbb{R}^d$ by
$$\Theta_i([x]) = \theta_i(x),\ x \in \Omega_i \qquad \Theta_j([y]) = \theta_j(y),\ y \in \Omega_j.$$
For $(i,j) \in K$, as the the τ_i's are injections we can define $\Theta_i \colon \tau_i(\Omega_i - \Omega_{ij}) \to \mathbb{R}^d$ and $\Theta_j \colon \tau_i(\Omega_j - \Omega_{ji}) \to \mathbb{R}^d$ by
$$\Theta_i([x]) = \theta_i(x),\ x \in \Omega_i - \Omega_{ij} \qquad \Theta_j([y]) = \theta_j(y),\ y \in \Omega_j - \Omega_{ji}.$$
It remains to define Θ_i on $\tau_i(\Omega_{ij})$ and Θ_j on $\tau_j(\Omega_{ji})$ in such a way that they agree on $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$. However, condition (C) in Definition 4.3 says that for all $x \in \Omega_{ij}$,
$$\theta_i(x) = \theta_j(\varphi_{ji}(x)).$$
Consequently, if we define Θ_i on $\tau_i(\Omega_{ij})$ and Θ_j on $\tau_j(\Omega_{ji})$ by
$$\Theta_i([x]) = \theta_i(x),\ x \in \Omega_{ij}, \qquad \Theta_j([y]) = \theta_j(y),\ y \in \Omega_{ji},$$
as $x \sim \varphi_{ji}(x)$, we have
$$\Theta_i([x]) = \theta_i(x) = \theta_j(\varphi_{ji}(x)) = \Theta_j([\varphi_{ji}(x)]) = \Theta_j([x]),$$
which means that Θ_i and Θ_j agree on $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$. But then, the functions, Θ_i, agree whenever their domains overlap and so, they patch to yield a function, Θ, with domain $M_{\mathcal{G}}$ and image M. By construction, $\theta_i = \Theta \circ \tau_i$ and as a manifold induced by $\mathcal{G}$ is a parametric pseudo-manifold, the last statement is obvious. □

The function, $\Theta \colon M_{\mathcal{G}} \to M$, given by Proposition 4.4 shows how the parametric pseudo-manifold, M, differs from the abstract manifold, $M_{\mathcal{G}}$. As we said before, a practical method for approximating 3D meshes based on parametric pseudo surfaces is described in Siqueira, Xu and Gallier [138].