Parametric Pseudo-Manifolds and Applications in Graphics

Luiz Velho
IMPA

Marcelo Siqueira
UFRN

Mini-Course, Summer 2009, IMPA

February 2-6

Schedule

February 2, 2009 (Monday), 1-3 PM

1. Overview of Manifolds
2. Applications in Graphics and Vision

February 3, 2009 (Tuesday), 1-3 PM

3. Constructing Manifolds
4. An Application: Surface Fitting

February 5, 2009 (Thursday), 1-3 PM

5. A Manifold-Based Construction for Surface Fitting

February 6, 2009 (Friday), 1-3 PM

6. Implementation Details

Overview of Manifolds

Lecture 1 - February 2, 2009 - 1-2 PM

Outline

Outline

- Manifolds: Brief History
- Informal definition
- Formal definition
- Example
  - The Sphere
- Conclusions

Origins of Manifolds

Origins of Manifolds

- Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.

Origins of Manifolds

- Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.

- In the early 1900's, Dehn, Heegaard, Veblen and Alexander routinely used the term **manifold**.

Origins of Manifolds

- Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.

- In a famous paper published in 1888, Dyck already uses the term **manifold** (in German).

- In the early 1900's, Dehn, Heegaard, Veblen and Alexander routinely used the term **manifold**.

Origins of Manifolds

- Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.

- In a famous paper published in 1888, Dyck already uses the term **manifold** (in German).

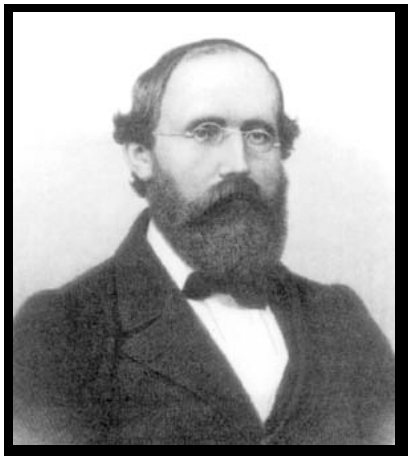
- In the early 1900's, Dehn, Heegaard, Veblen and Alexander routinely used the term **manifold**.

- Hermann Weyl was among the first to give a rigorous definition (1913).

Keys Contributors to the notion of manifold:

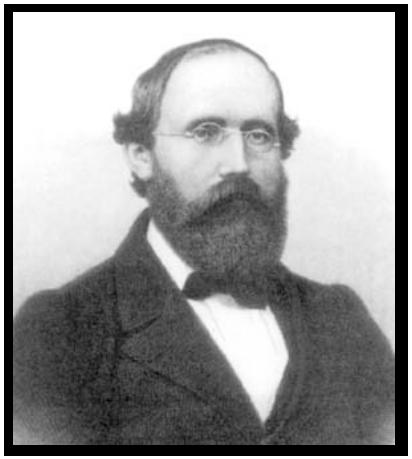
Keys Contributors to the notion of manifold:

Georg Friedrich Bernhard Riemann
1826-1866



Keys Contributors to the notion of manifold:

Georg Friedrich Bernhard Riemann
1826-1866



Hermann Klaus Hugo Weyl
1885-1955

Keys Contributors to the notion of manifold:

Keys Contributors to the notion of manifold:

Hermann Weyl (again)

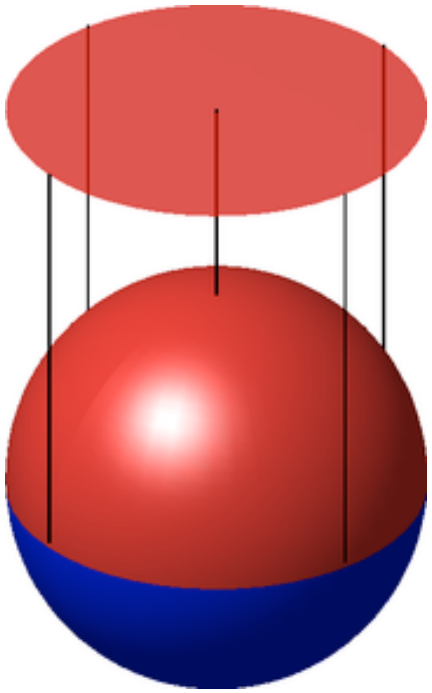
Keys Contributors to the notion of manifold:

Hermann Weyl (again)

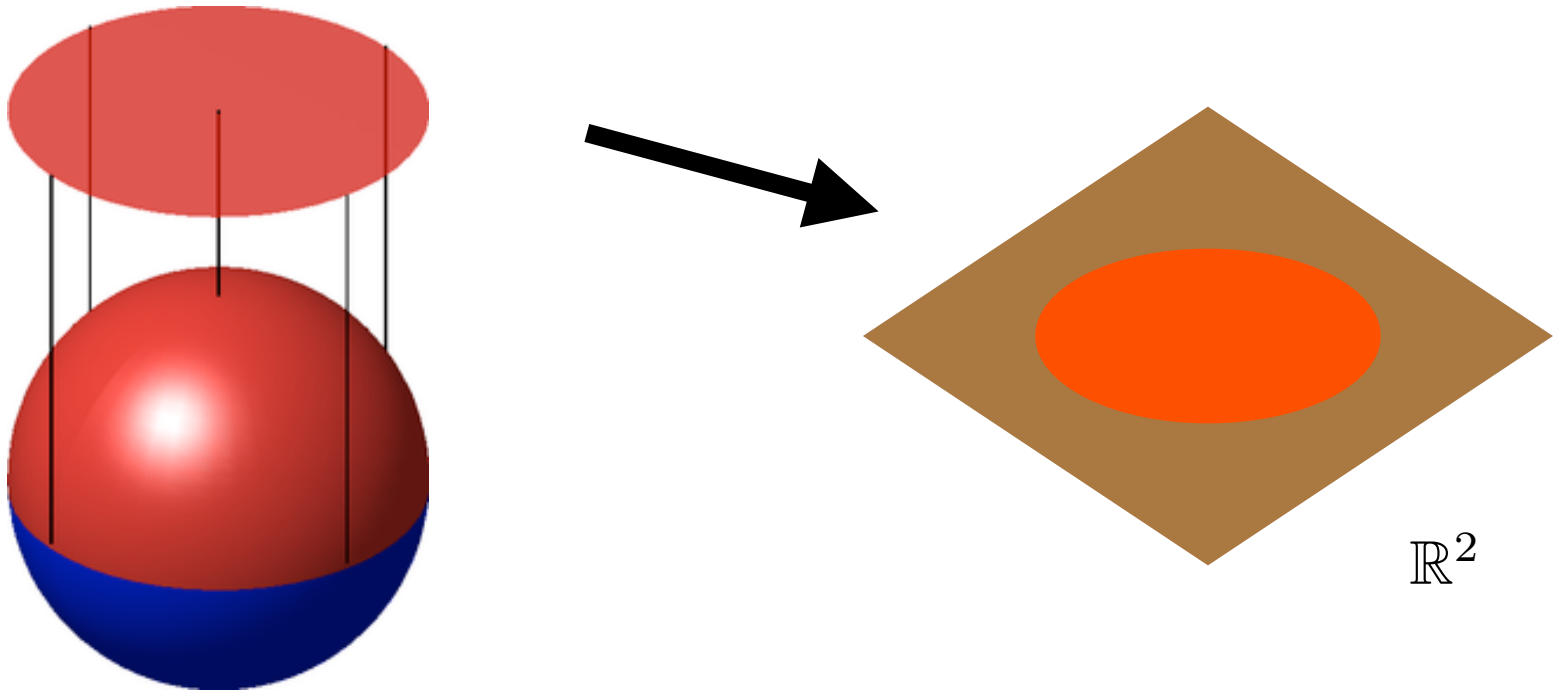
Hassler Whitney
1907-1989

Why we need Manifolds?

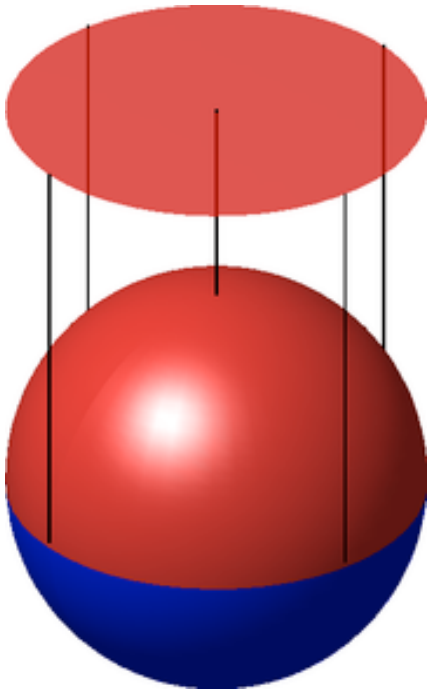
Why we need Manifolds?

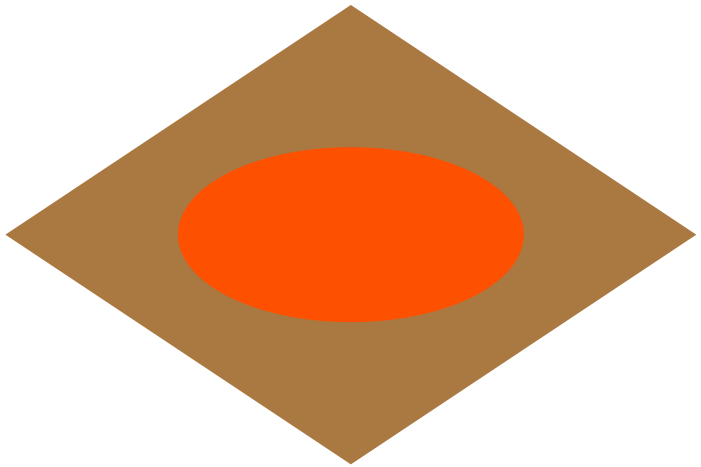


Why we need Manifolds?



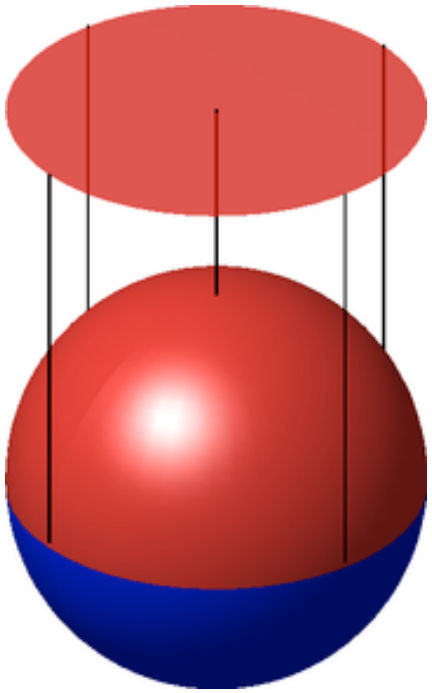
Manifolds: Informal Definition

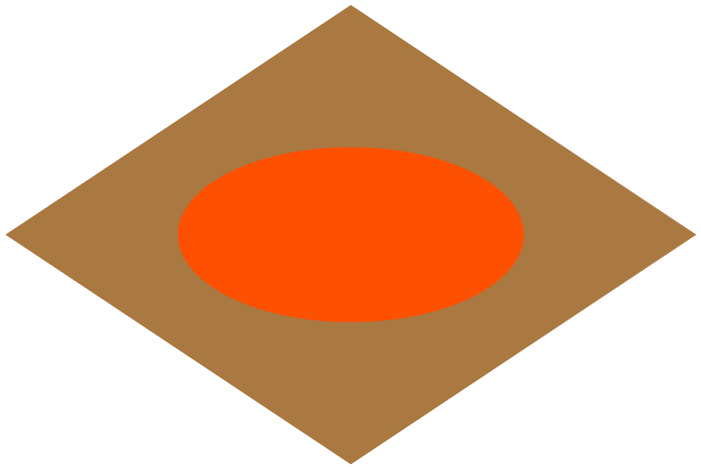




Manifolds: Informal Definition

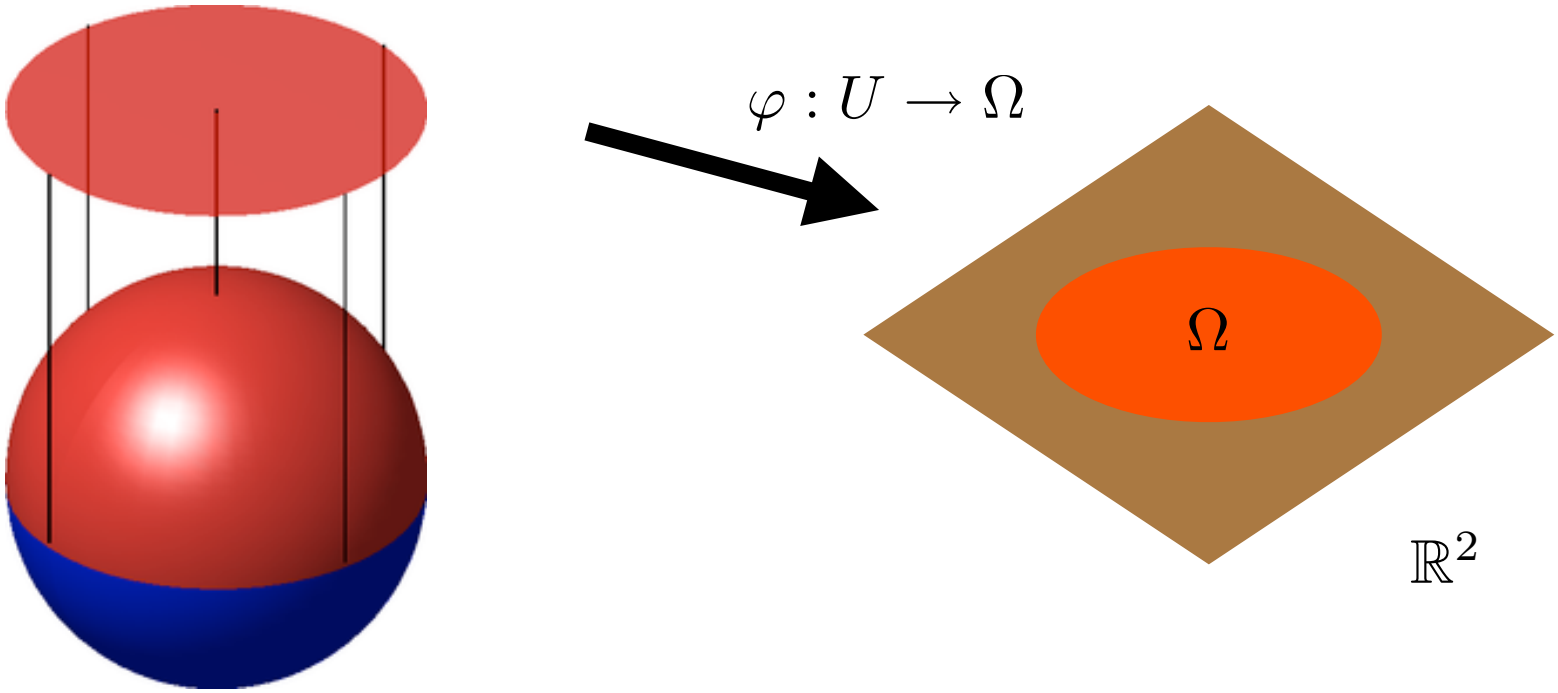
- A manifold is a topological space with an open cover so that every open set in this cover "looks" like an open subset of $\mathbb{R}^n$.





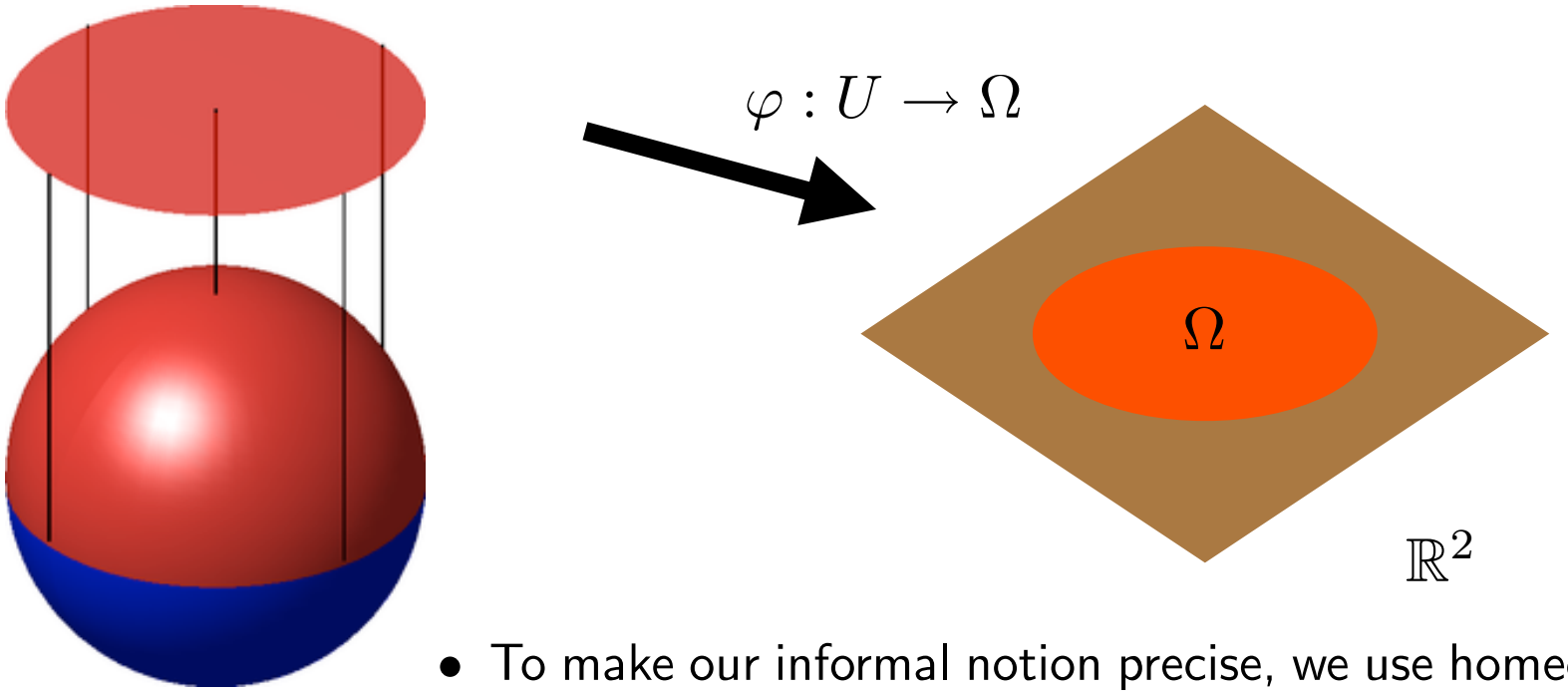
Manifolds: Informal Definition

- A manifold is a topological space with an open cover so that every open set in this cover "looks" like an open subset of $\mathbb{R}^n$.



Manifolds: Informal Definition

- A manifold is a topological space with an open cover so that every open set in this cover "looks" like an open subset of $\mathbb{R}^n$.



- To make our informal notion precise, we use homeomorphisms, $\varphi : U \to \Omega$, where $\Omega \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.

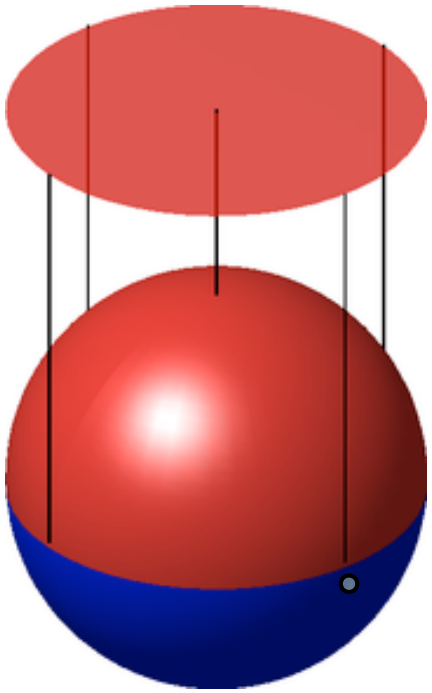
Manifolds: Informal Definition

Manifolds: Informal Definition

- We also want to be able "to do calculus" on our manifolds. For this we need some conditions on **overlaps** of open sets.

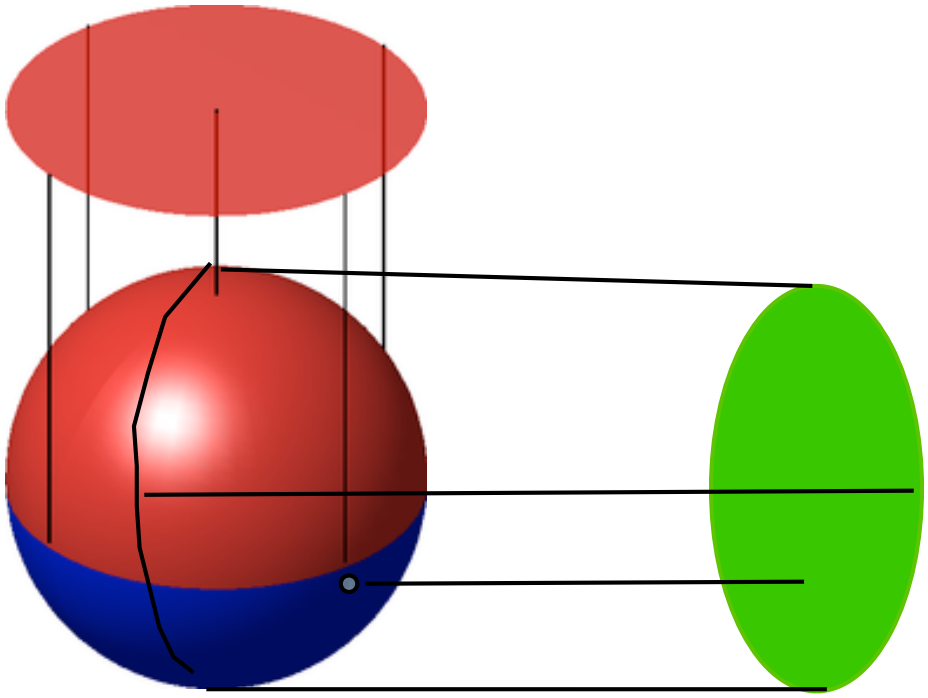
Manifolds: Informal Definition

- We also want to be able "to do calculus" on our manifolds. For this we need some conditions on **overlaps** of open sets.



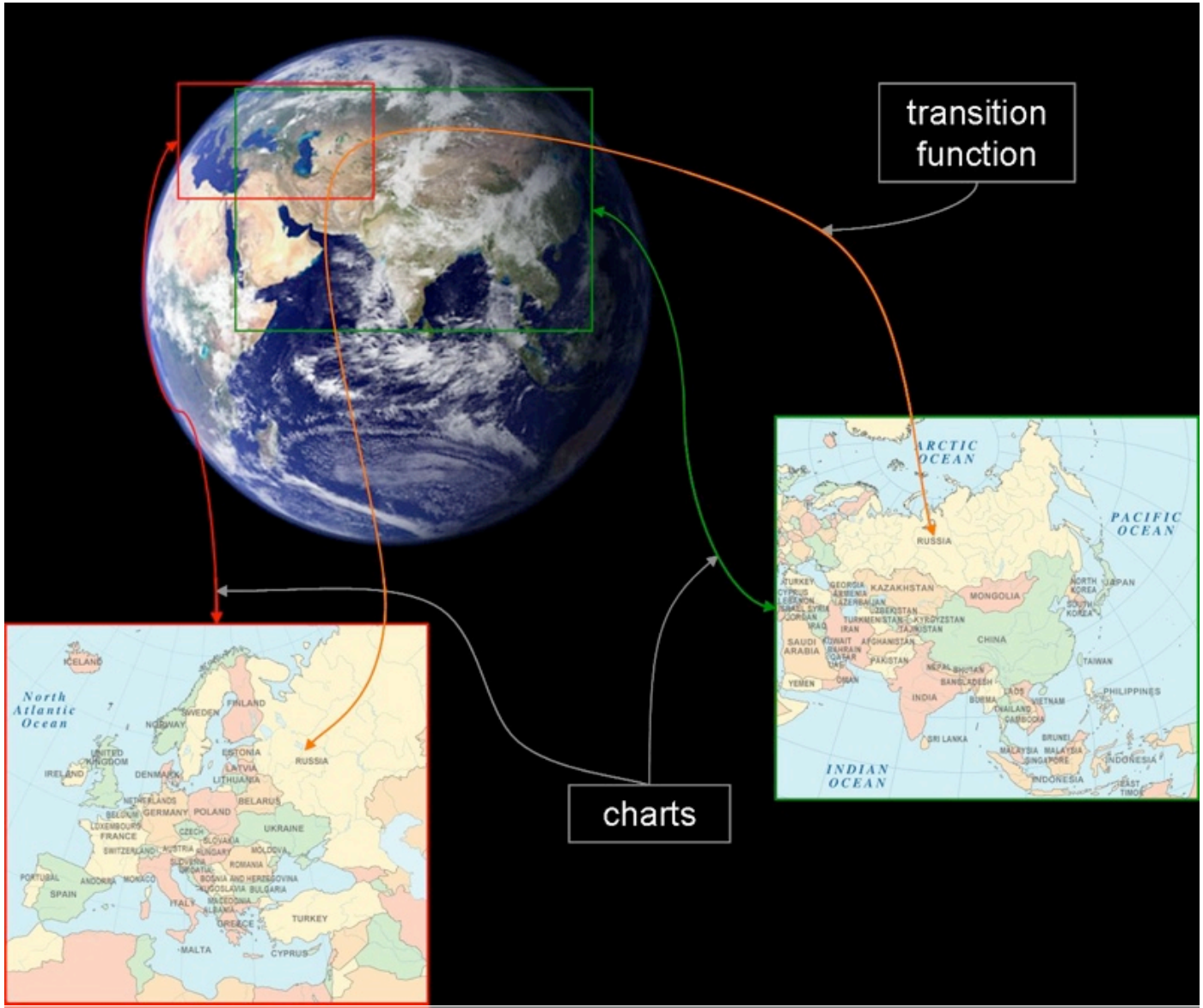
Manifolds: Informal Definition

- We also want to be able "to do calculus" on our manifolds. For this we need some conditions on **overlaps** of open sets.



Manifold: An Intuitive Picture

Manifold: An Intuitive Picture



Manifolds: Informal Definition

Manifolds: Informal Definition

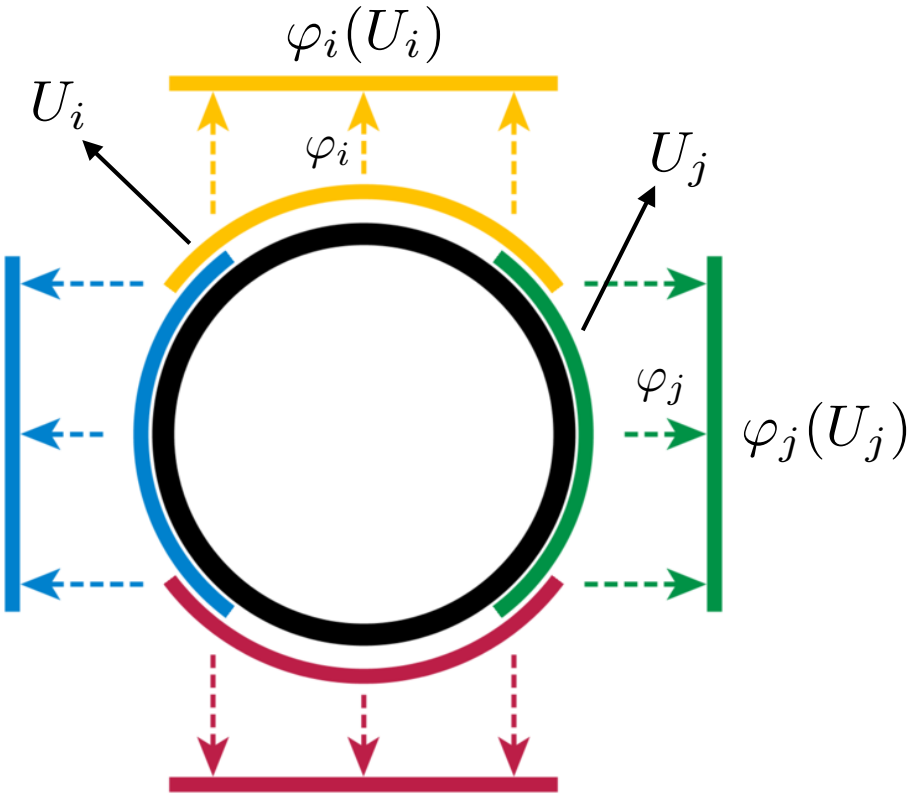
- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)\,.$$

Manifolds: Informal Definition

- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

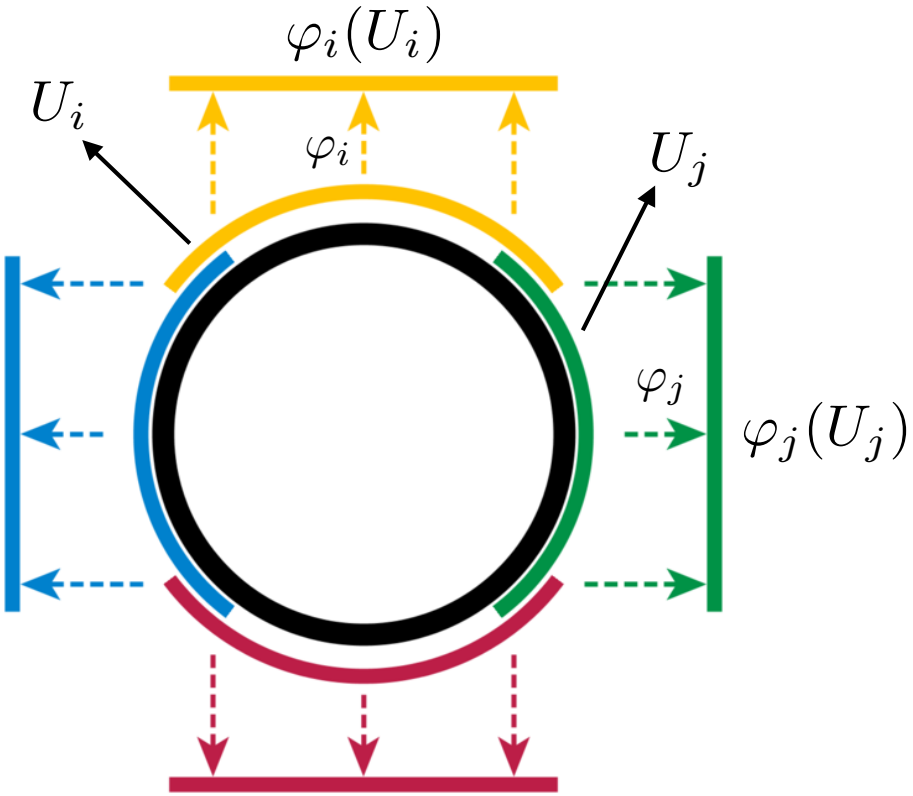
$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)\,.$$

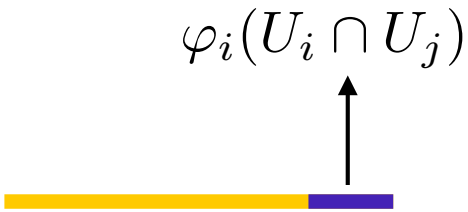


Manifolds: Informal Definition

- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)\,.$$

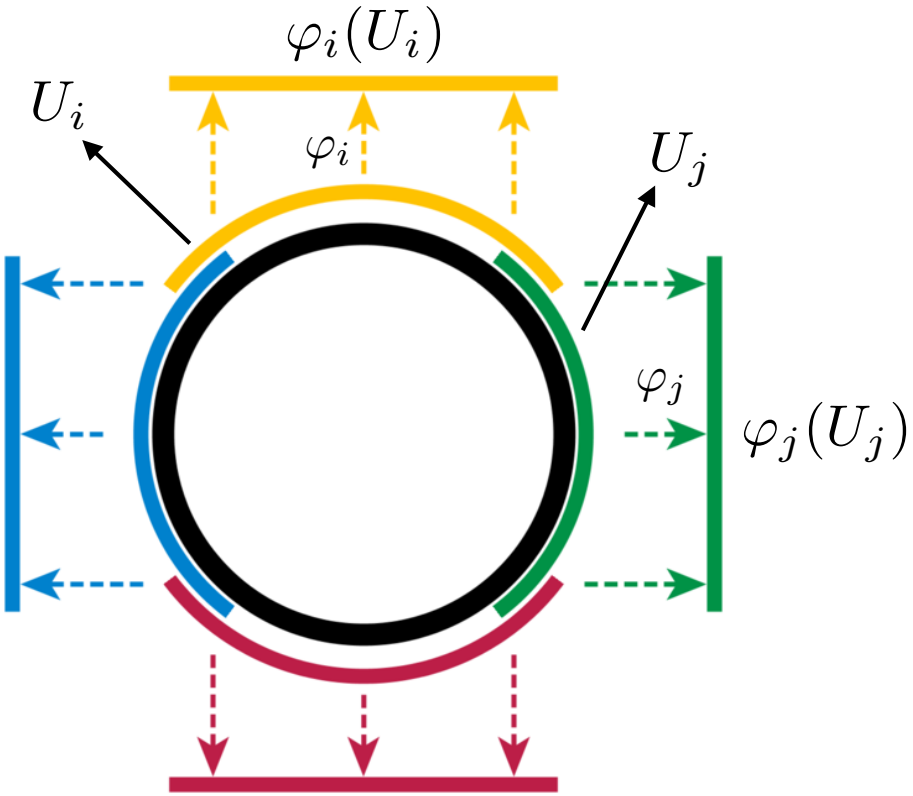


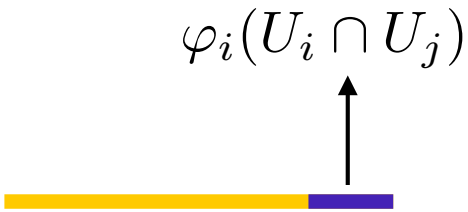


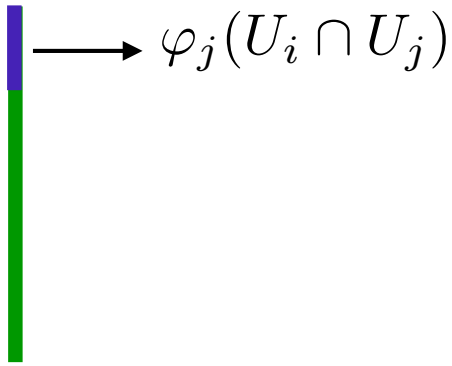
Manifolds: Informal Definition

- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)\,.$$



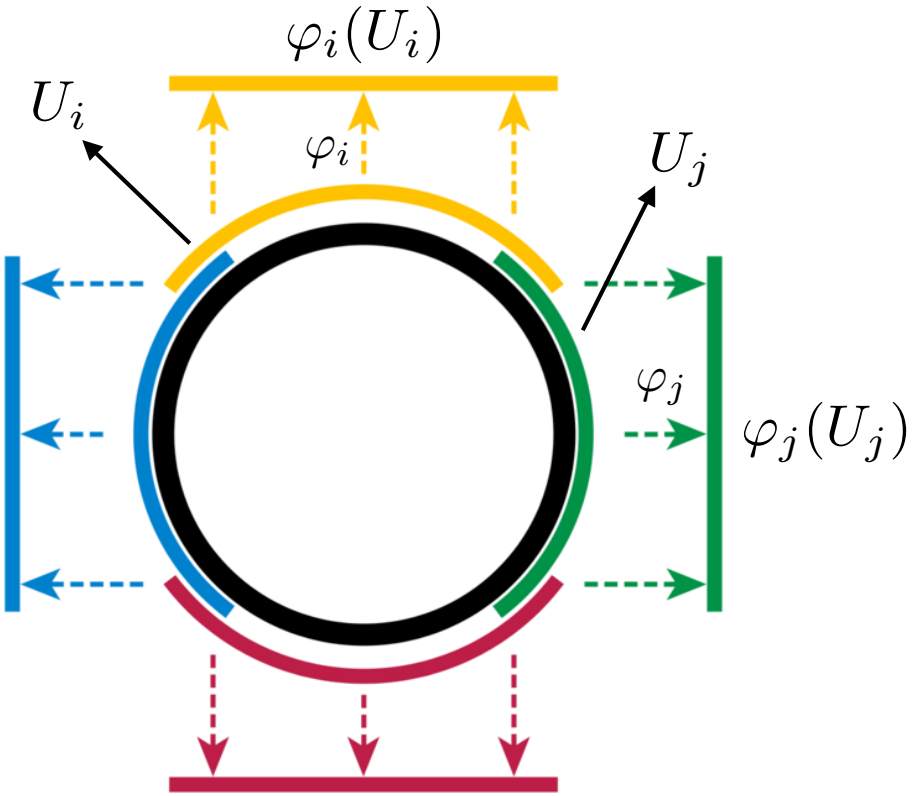


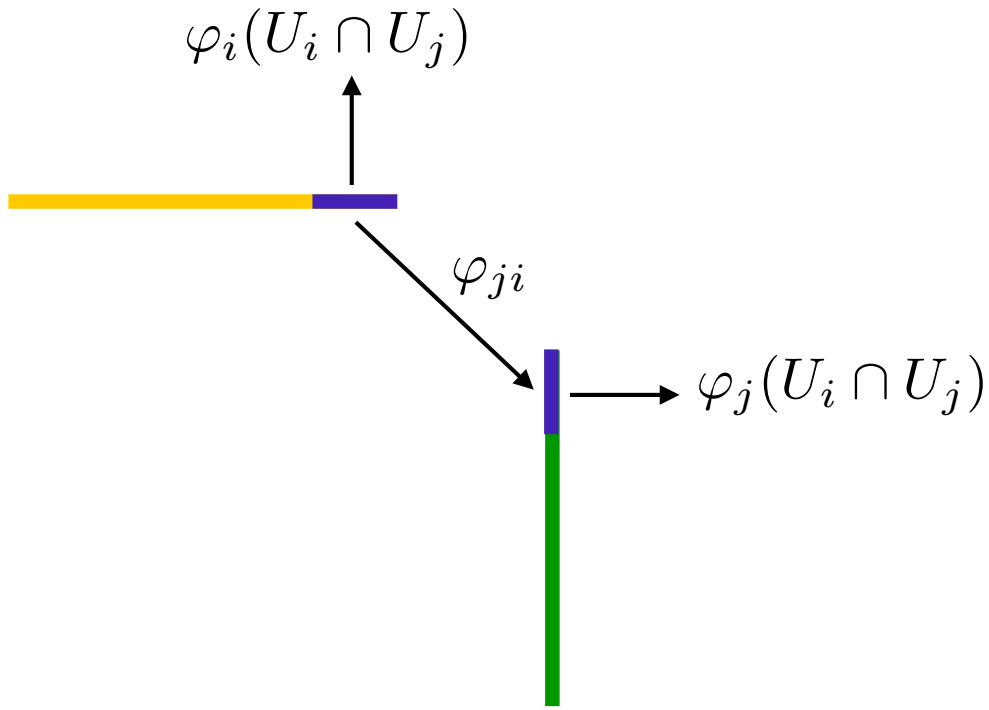


Manifolds: Informal Definition

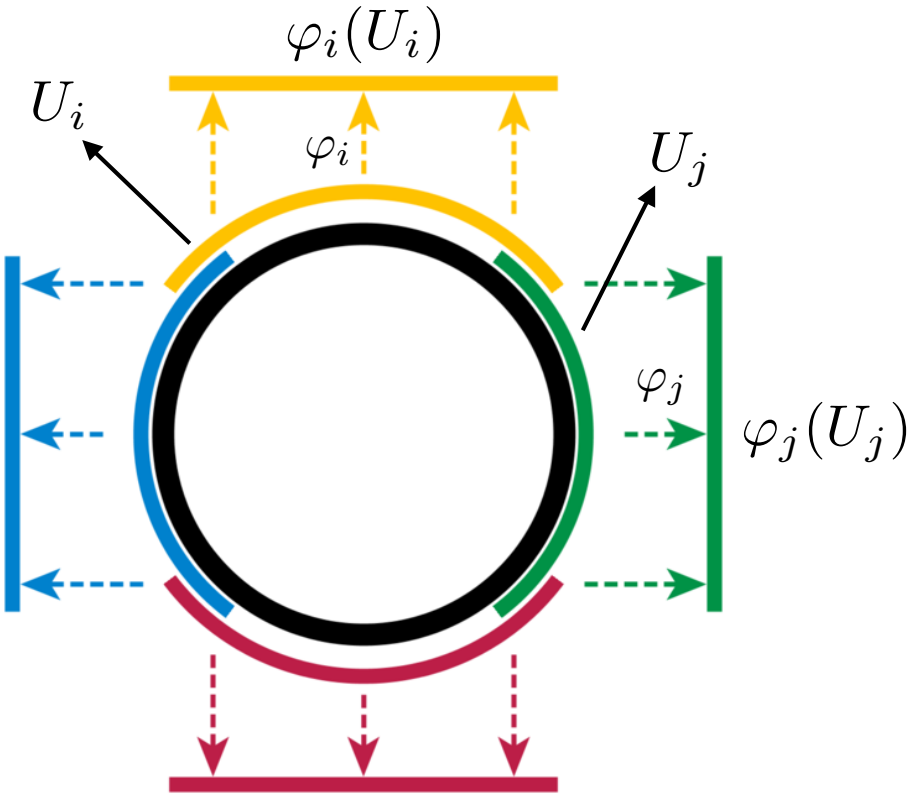
- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

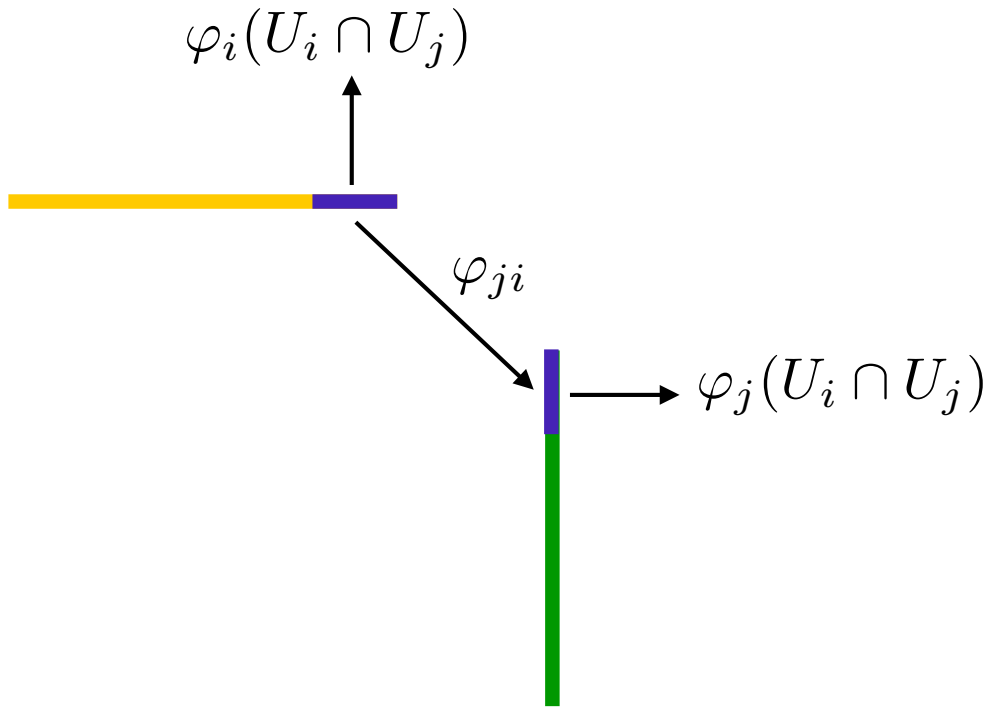
$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)\,.$$





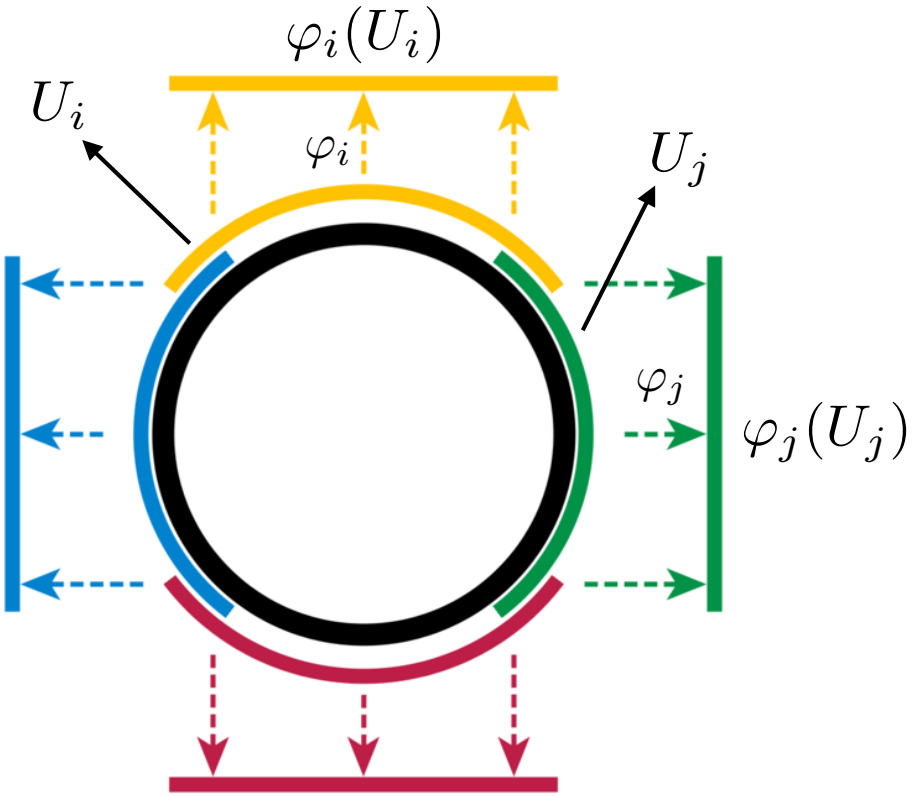
Manifolds: Informal Definition

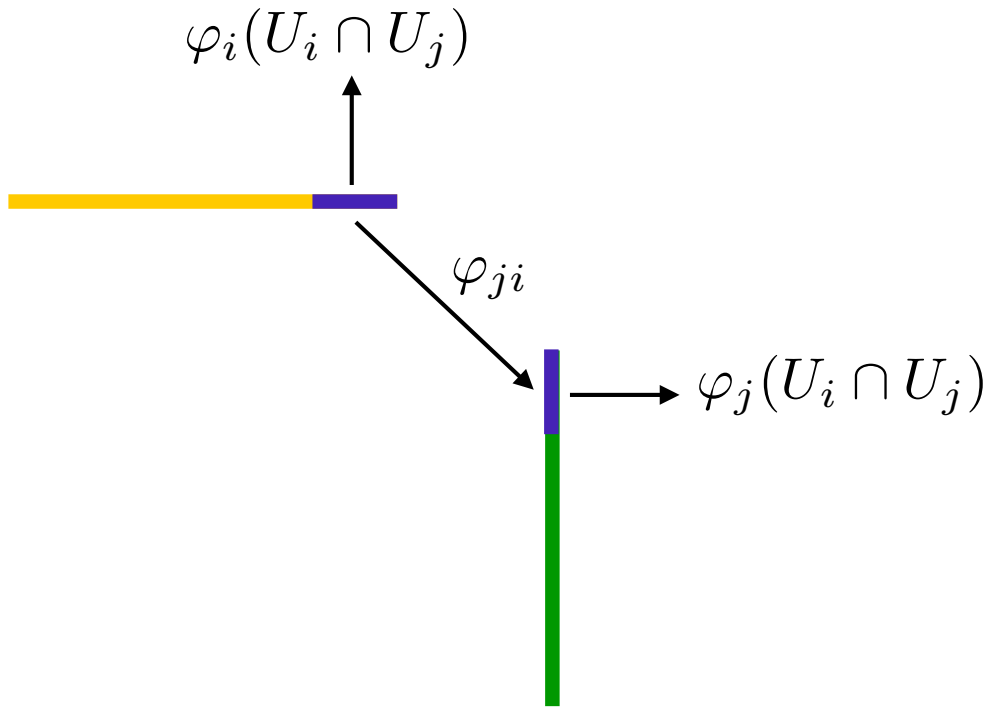




Manifolds: Informal Definition

- This is a map between two open subsets of $\mathbb{R}^n$ and we require it possess a certain amount of **smoothness**.





Manifolds: Formal Definition

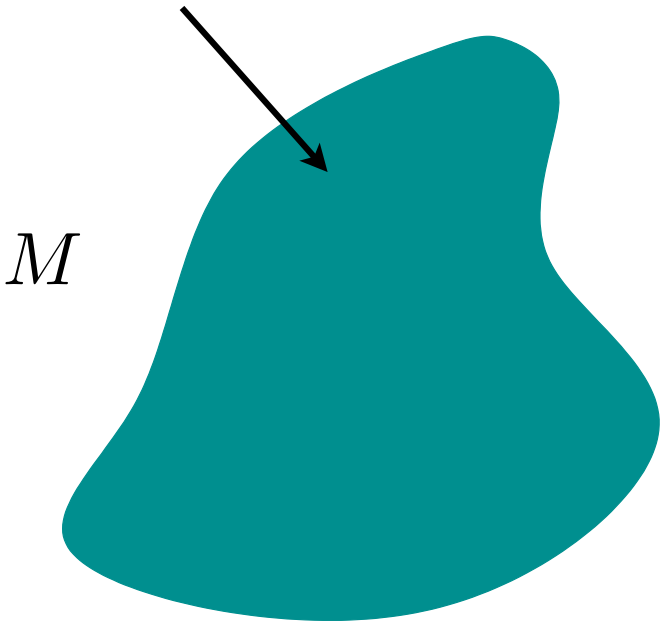
Manifolds: Formal Definition

Recall the definition of a manifold...

Manifolds: Formal Definition

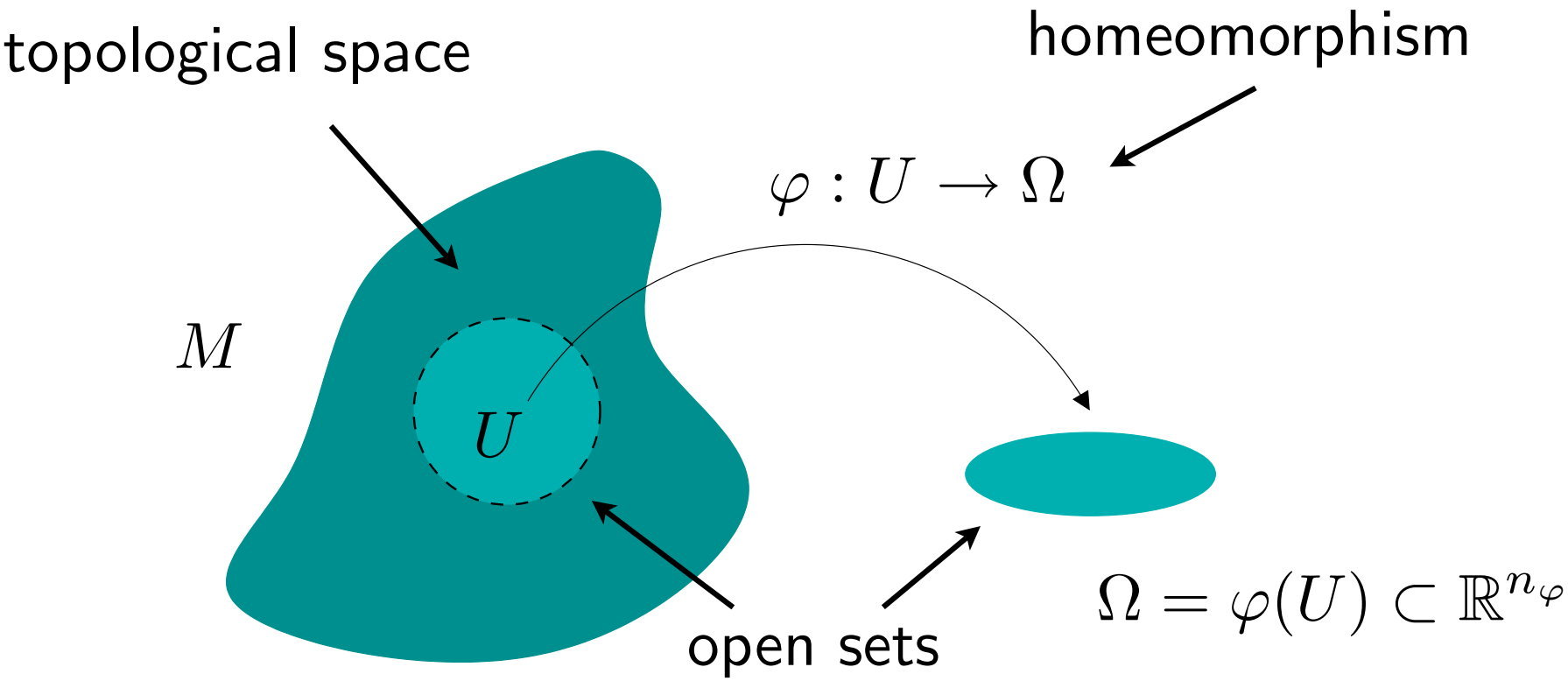
Recall the definition of a manifold...

topological space



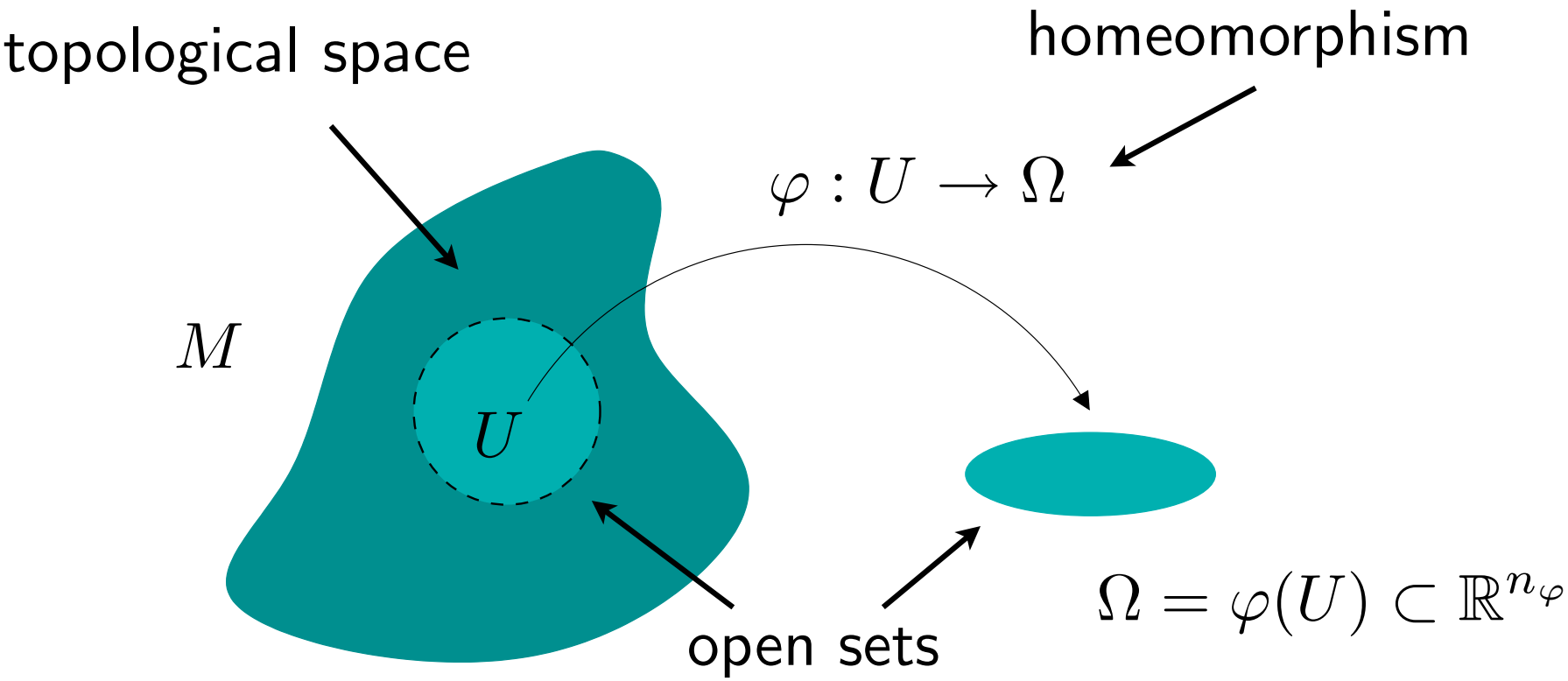
Manifolds: Formal Definition

Recall the definition of a manifold...



Manifolds: Formal Definition

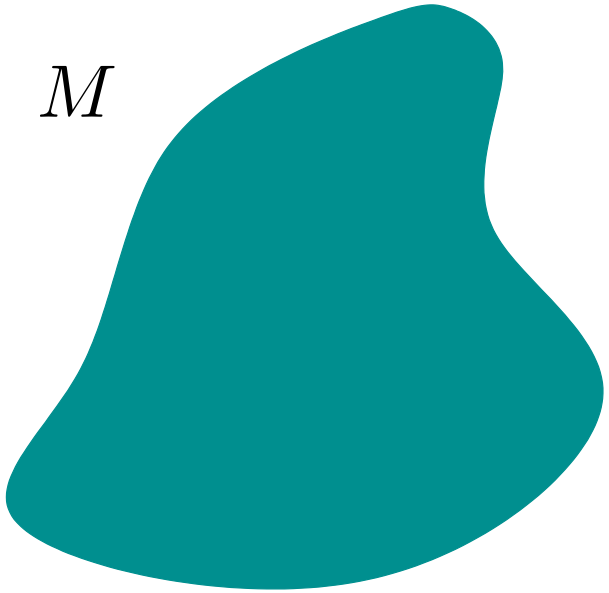
Recall the definition of a manifold...



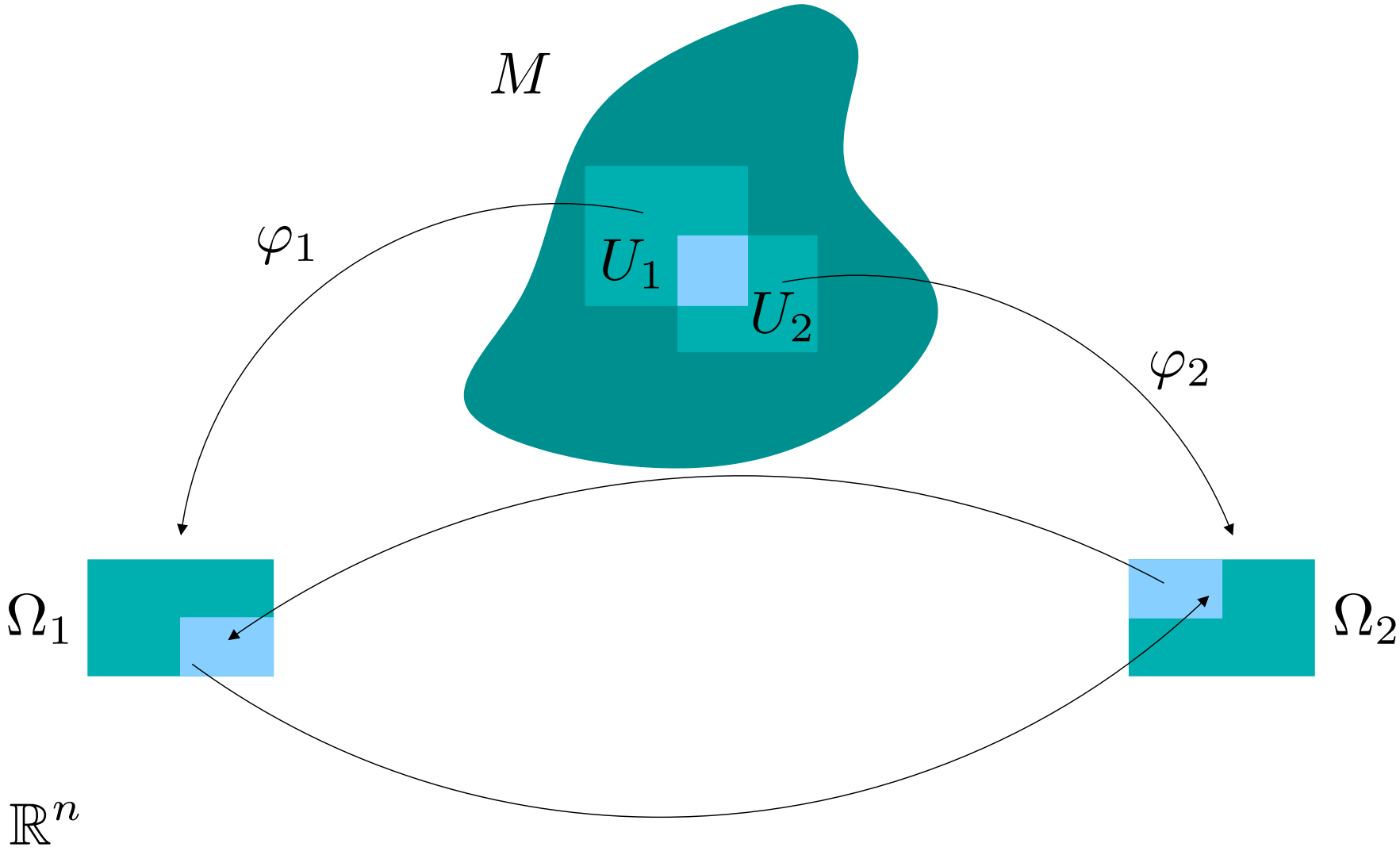
(U, φ) is called a **chart**.

Manifolds: Formal Definition

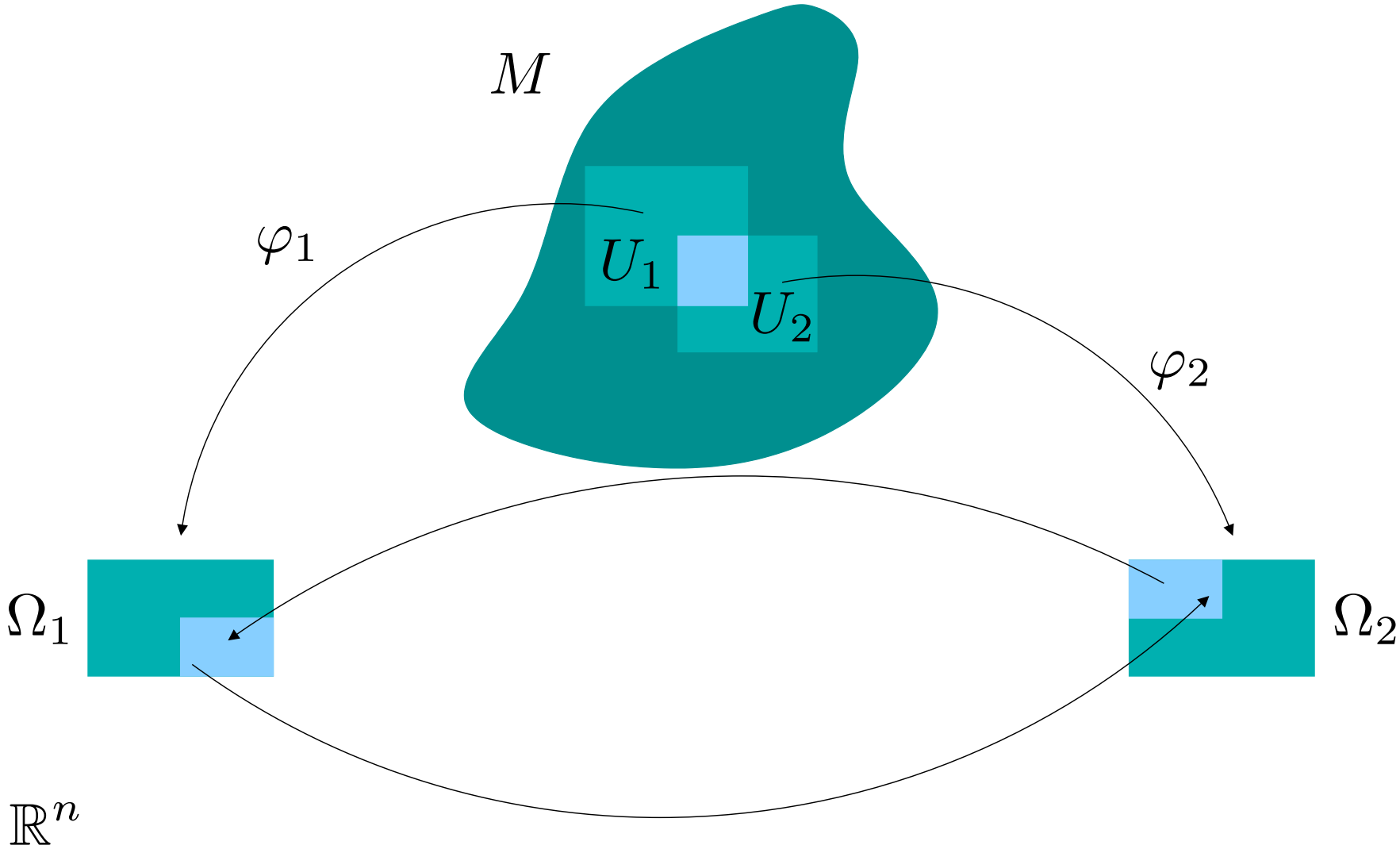
Manifolds: Formal Definition



Manifolds: Formal Definition

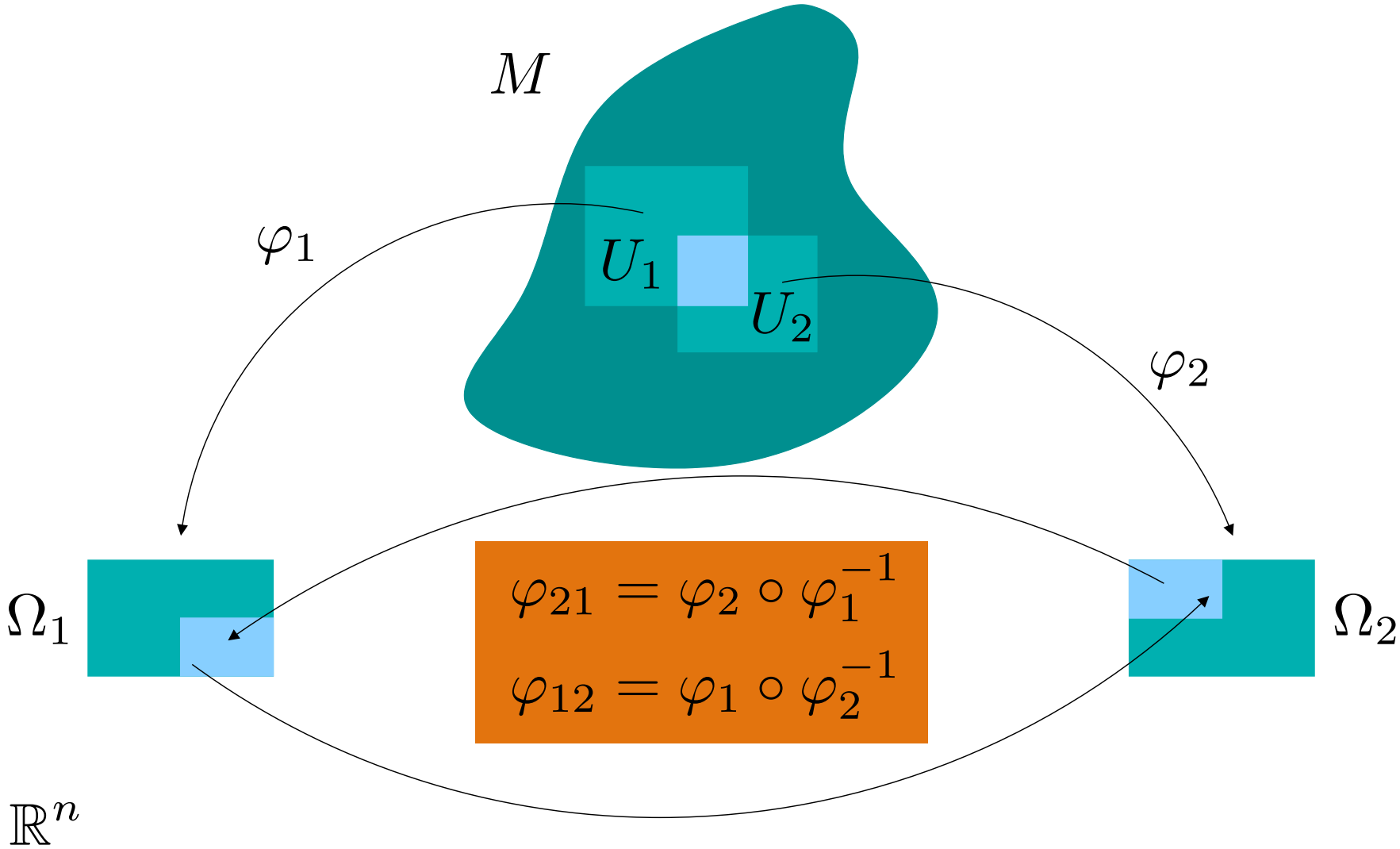


Manifolds: Formal Definition



$$\varphi_{21} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$
$$\varphi_{12} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$$

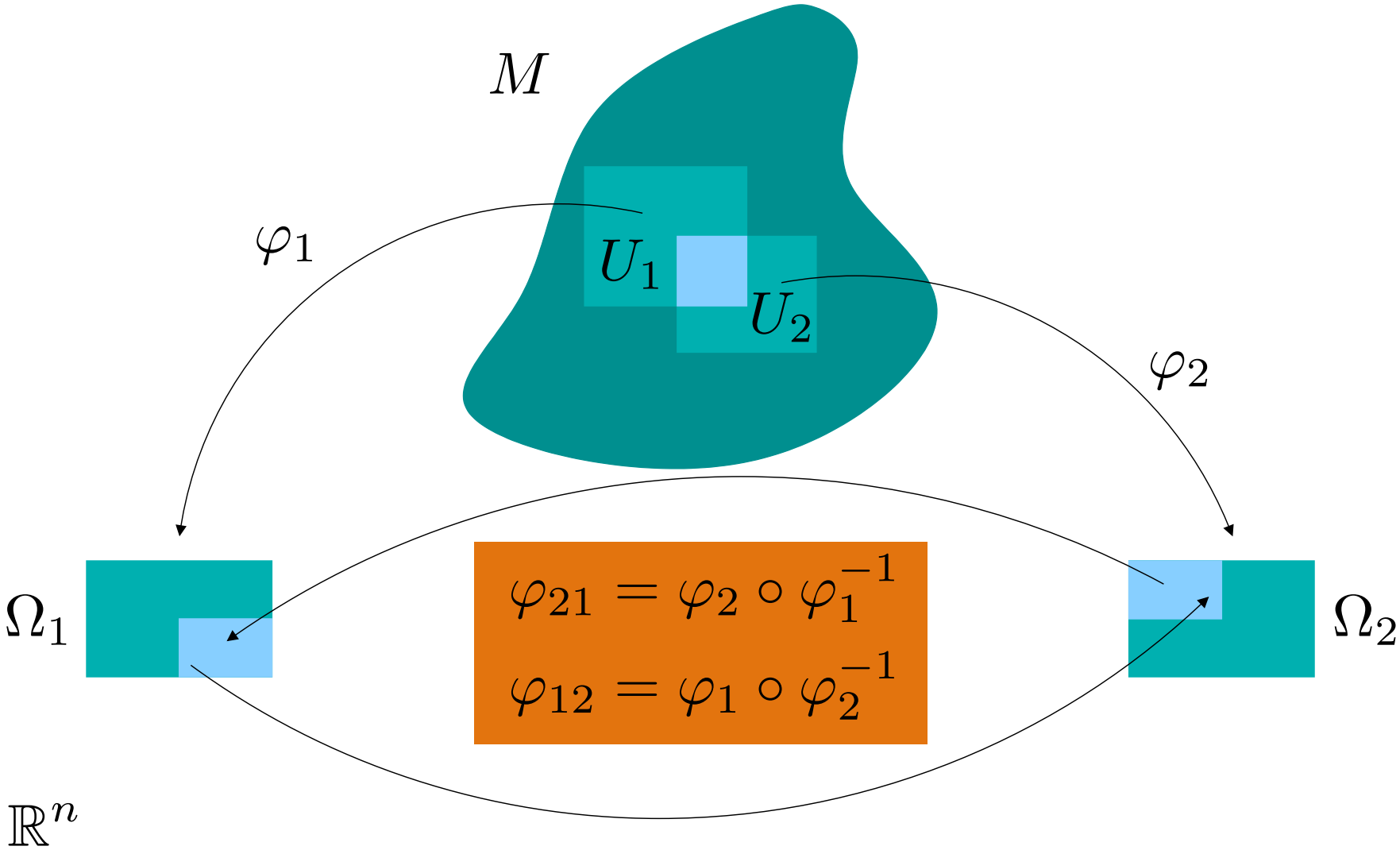
Manifolds: Formal Definition



$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$$
$$\varphi_{12} = \varphi_1 \circ \varphi_2^{-1}$$

$$\varphi_{21} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$
$$\varphi_{12} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$$

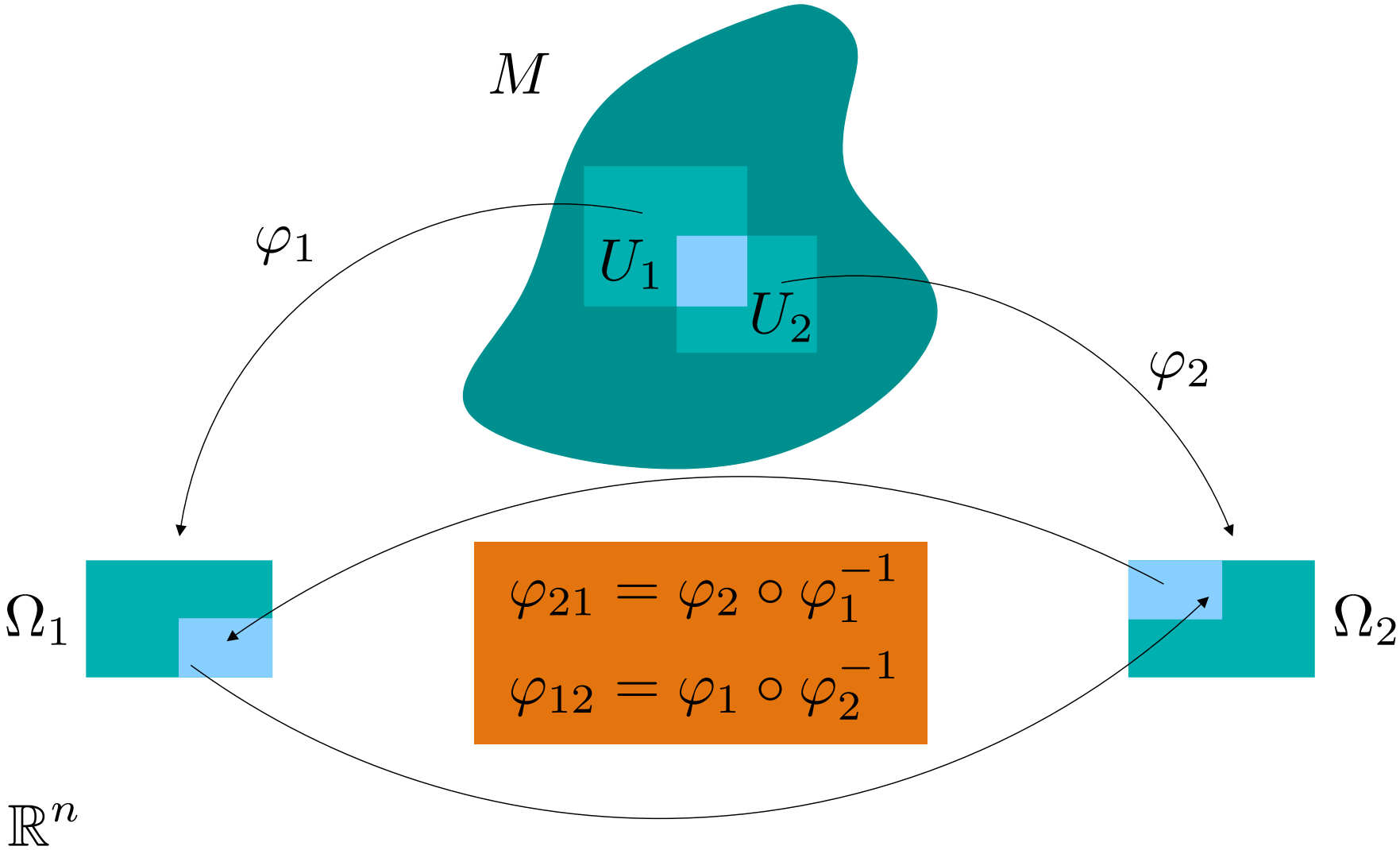
Manifolds: Formal Definition



$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$$

$$\varphi_{12} = \varphi_1 \circ \varphi_2^{-1}$$

Manifolds: Formal Definition



$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$$

$$\varphi_{12} = \varphi_1 \circ \varphi_2^{-1}$$

φ_{21} and φ_{12} are the **transition functions**.

Manifolds: Formal Definition

Manifolds: Formal Definition

A C^k n-**atlas** is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

Manifolds: Formal Definition

A C^k n-**atlas** is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i.

Manifolds: Formal Definition

A C^k n-**atlas** is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i.

(2) $M = \bigcup_{i \in I} U_i$.

Manifolds: Formal Definition

A C^k n-**atlas** is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

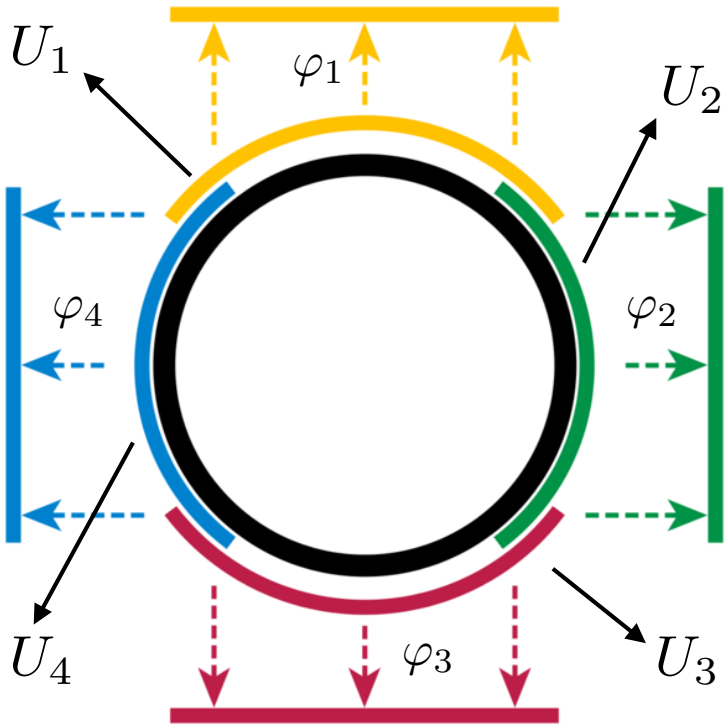
(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i.

(2) $M = \bigcup_{i \in I} U_i$.

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition function φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism.

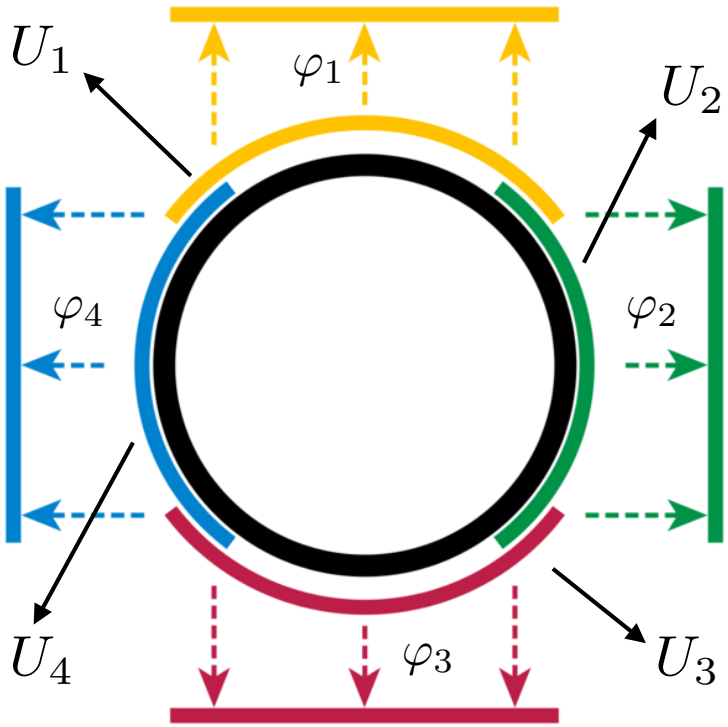
Manifolds: Formal Definition

Manifolds: Formal Definition



Atlas: $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$

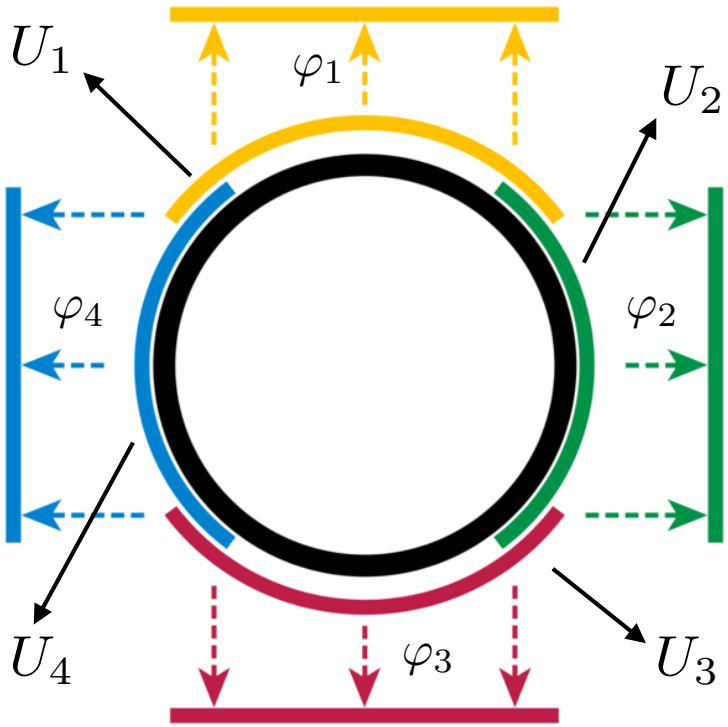
Manifolds: Formal Definition



$$M = \bigcup_{i=1}^{4} U_i$$

Atlas: $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$

Manifolds: Formal Definition



$$M = \bigcup_{i=1}^{4} U_i$$

φ_i is a C^k diffeomorphism

Atlas: $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$

Manifolds: Formal Definition

Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but...

Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but...

- there may be many choice of atlases;

Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but...

- there may be many choice of atlases;
- we get around this problem by defining a notion of atlas compatibility;

Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but...

- there may be many choice of atlases;
- we get around this problem by defining a notion of atlas compatibility;
- this notion induces an equivalence relation of atlases on M;

Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but...

- there may be many choice of atlases;
- we get around this problem by defining a notion of atlas compatibility;
- this notion induces an equivalence relation of atlases on M;
- the set of all charts compatible with a given atlas is a maximum atlas in its class.

Examples

Examples

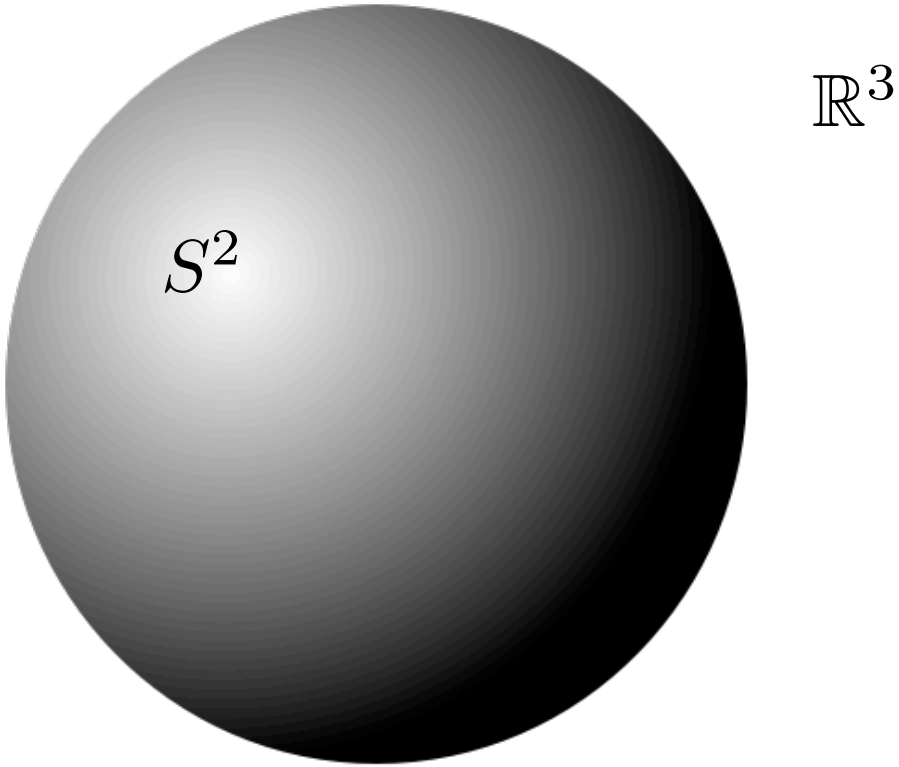
- The sphere

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

Examples

- The sphere

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}\,.$$



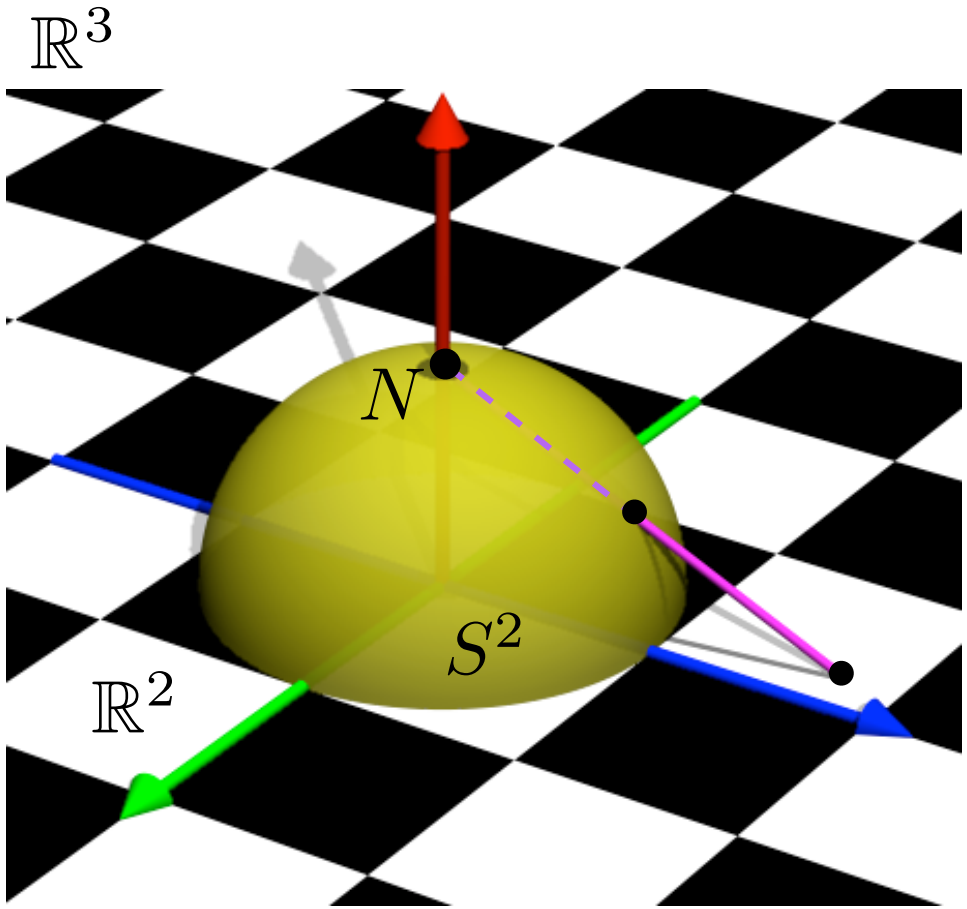
Examples

Examples

- We use stereographic projection from the north pole . . .

Examples

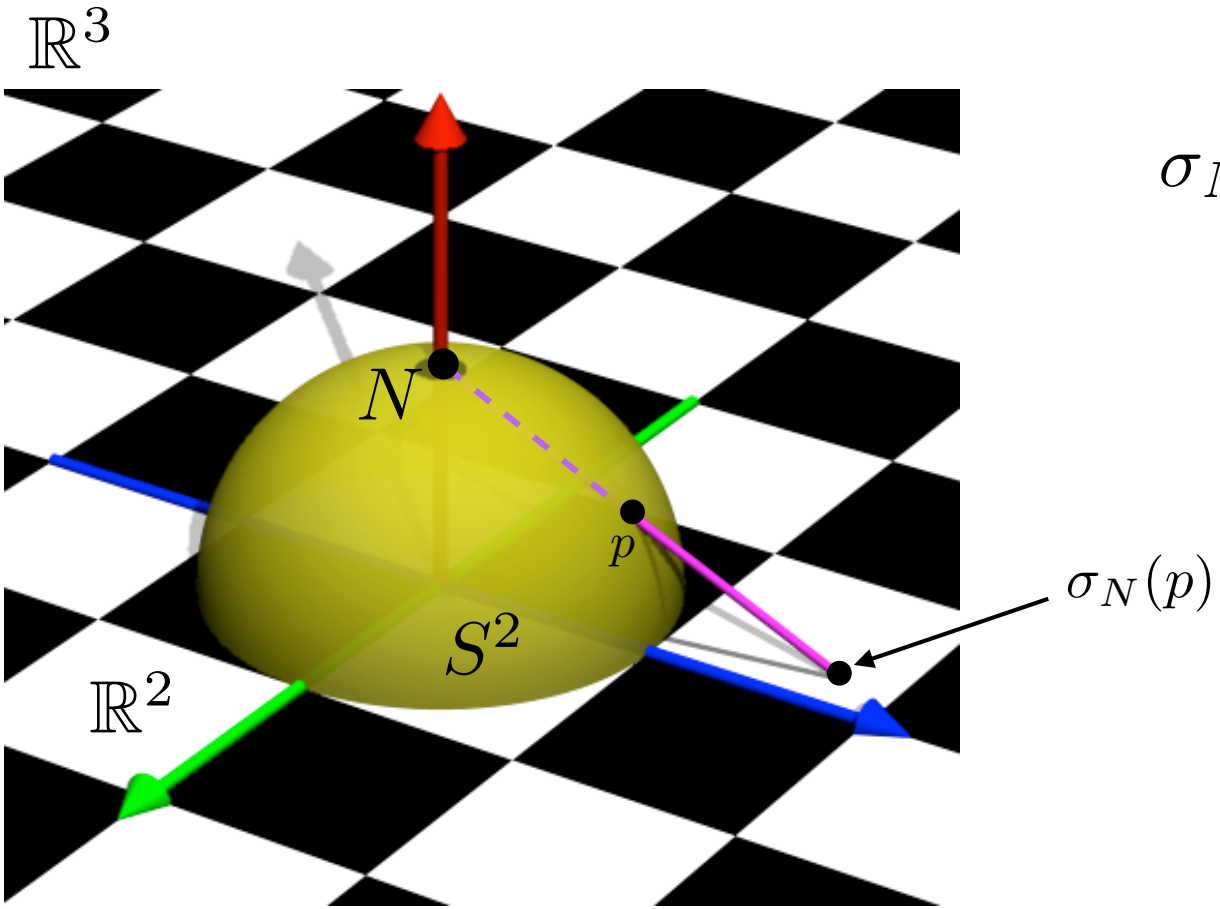
- We use stereographic projection from the north pole . . .



$$\sigma_N \colon S^n - \{N\} \longrightarrow \mathbb{R}^n$$

Examples

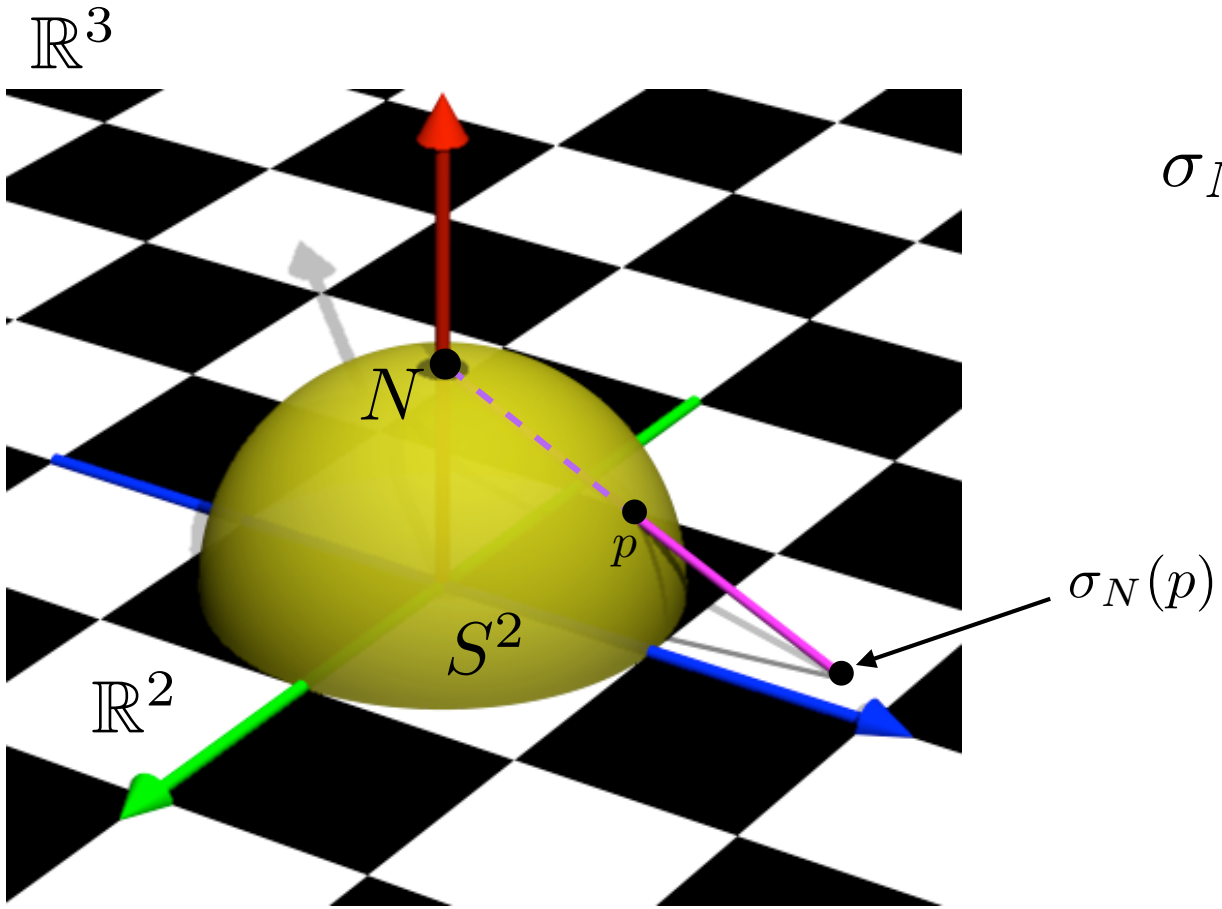
- We use stereographic projection from the north pole . . .



$$\sigma_N \colon S^n - \{N\} \longrightarrow \mathbb{R}^n$$

Examples

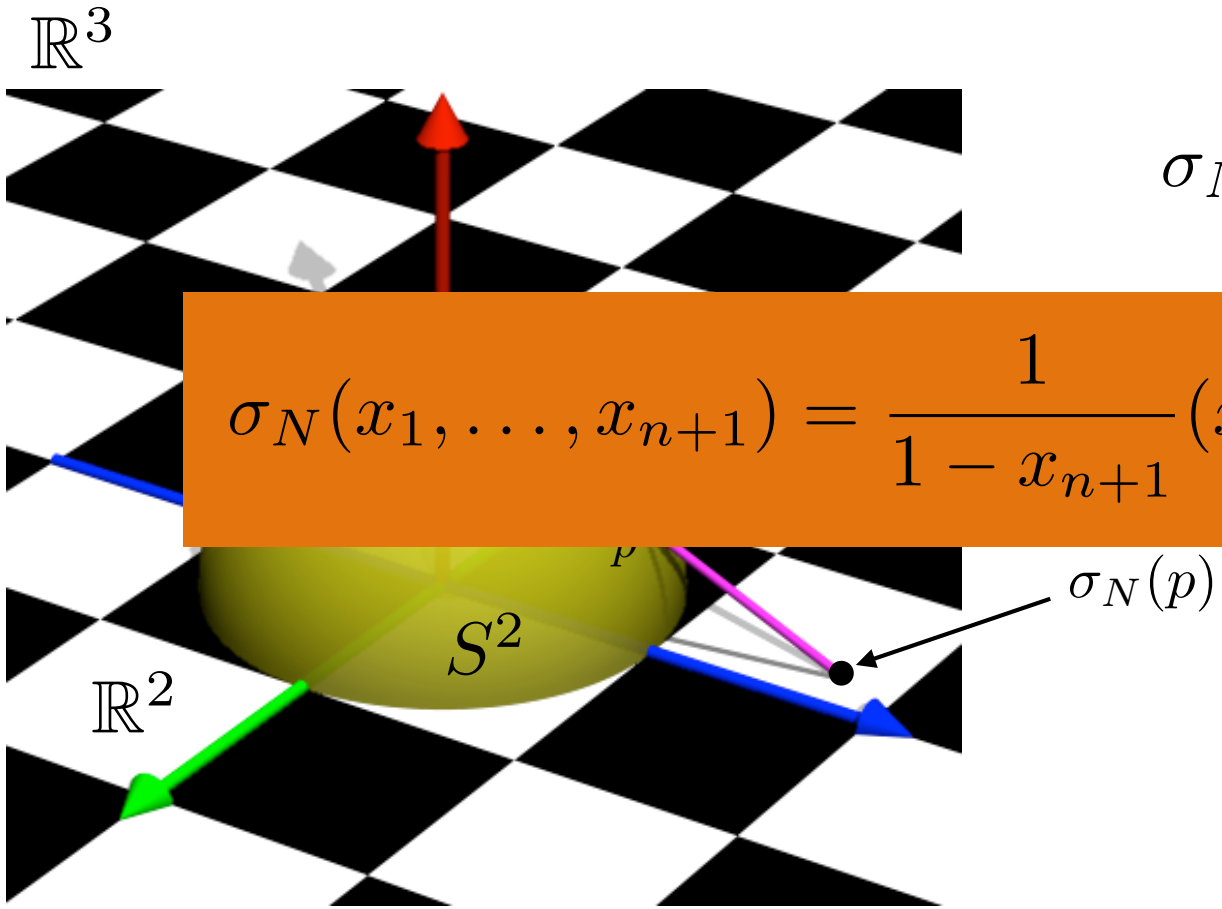
- We use stereographic projection from the north pole . . .



$$\sigma_N \colon S^n - \{N\} \longrightarrow \mathbb{R}^n$$

Examples

- We use stereographic projection from the north pole . . .



$$\sigma_N \colon S^n - \{N\} \longrightarrow \mathbb{R}^n$$

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

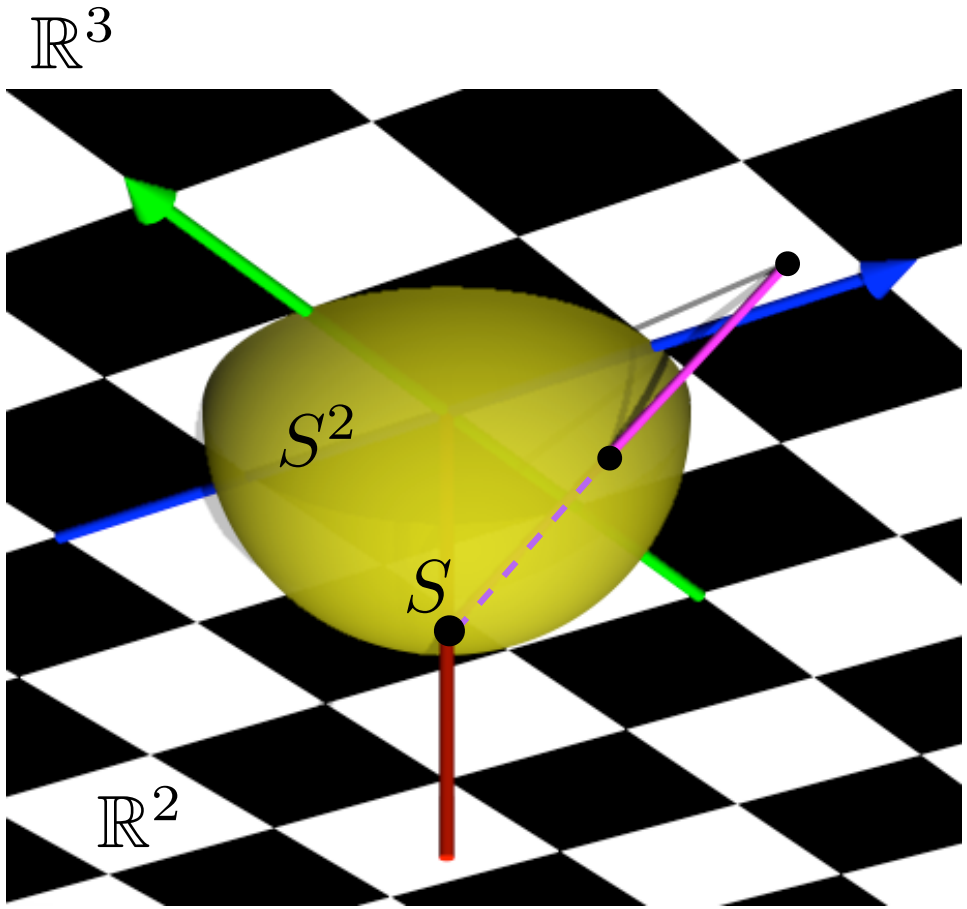
Examples

Examples

and from the south pole:

Examples

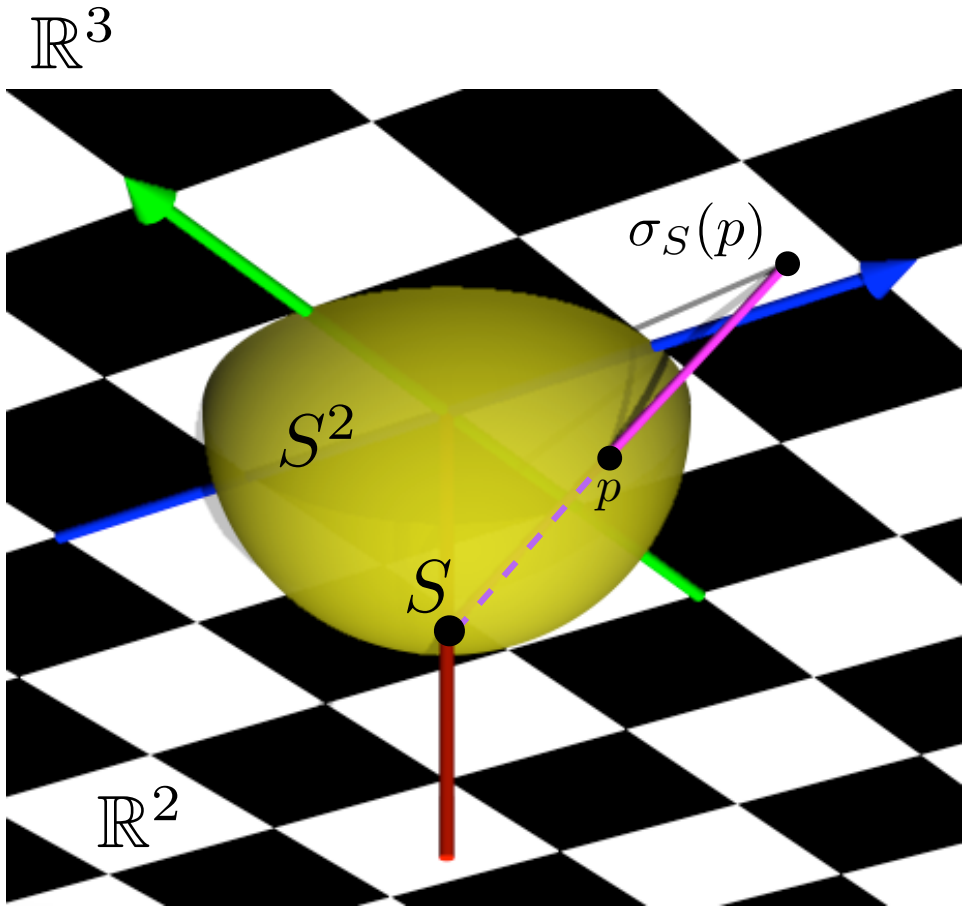
and from the south pole:



$$\sigma_S : S^n - \{S\} \longrightarrow \mathbb{R}^n$$

Examples

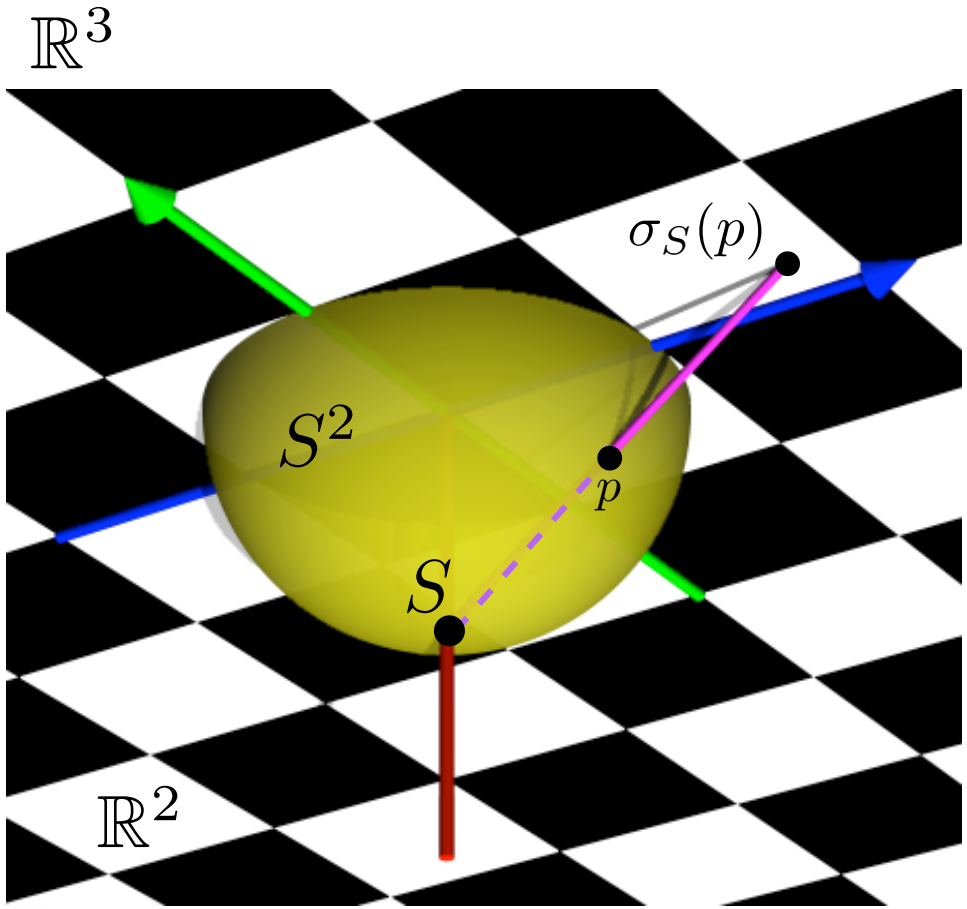
and from the south pole:



$$\sigma_S : S^n - \{S\} \longrightarrow \mathbb{R}^n$$

Examples

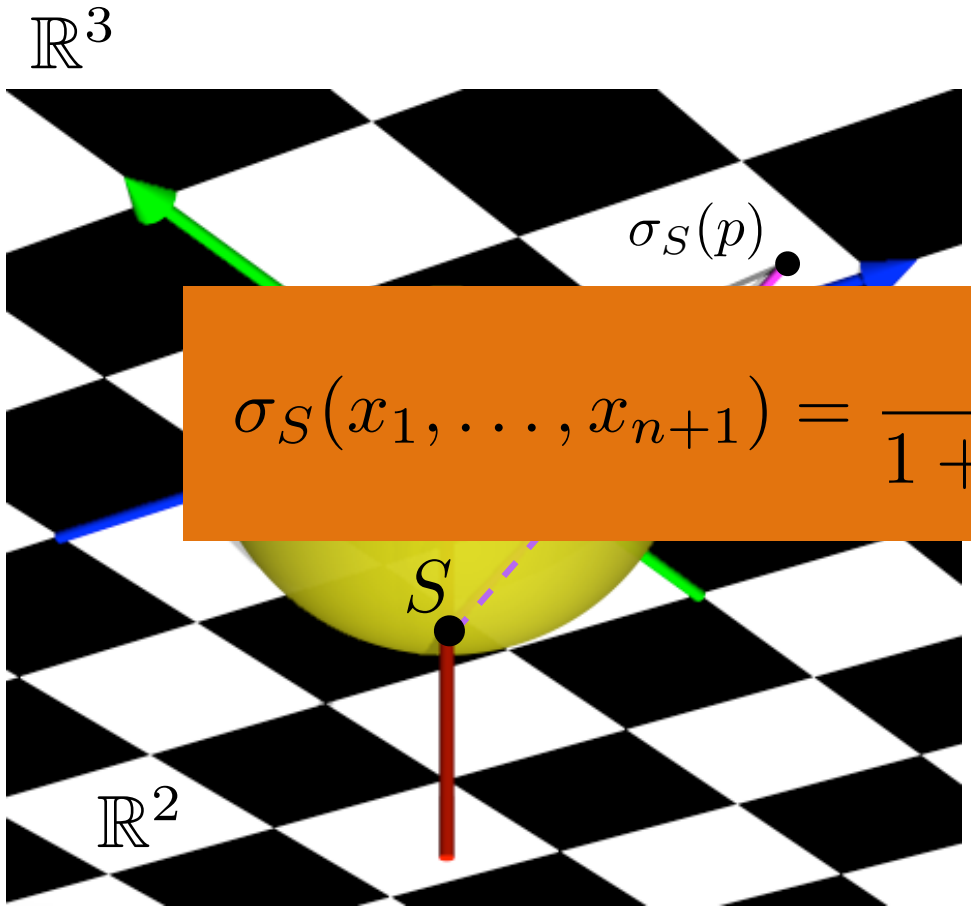
and from the south pole:



$$\sigma_S : S^n - \{S\} \longrightarrow \mathbb{R}^n$$

Examples

and from the south pole:



$$\sigma_S : S^n - \{S\} \longrightarrow \mathbb{R}^n$$

$$\sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n)$$

Examples

Examples

- Inverse stereographic projections:

Examples

- Inverse stereographic projections:

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

Examples

Examples

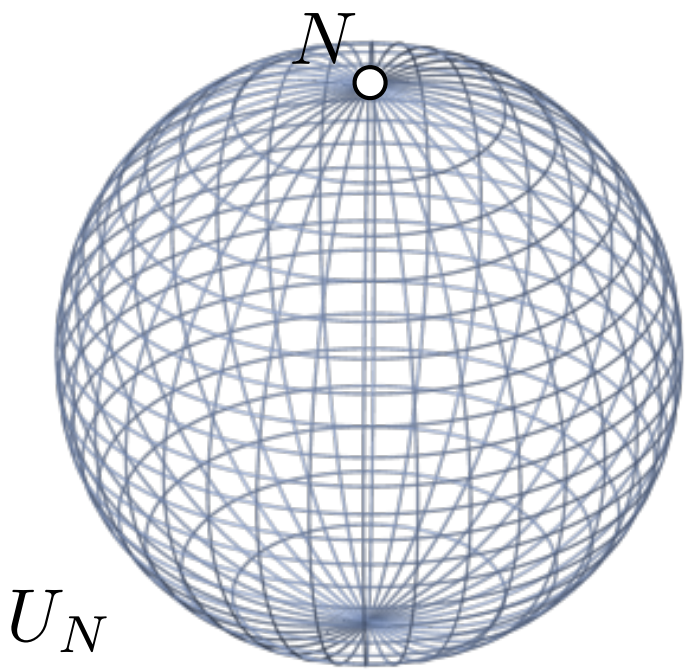
- Consider the open cover consisting of

$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$

Examples

- Consider the open cover consisting of

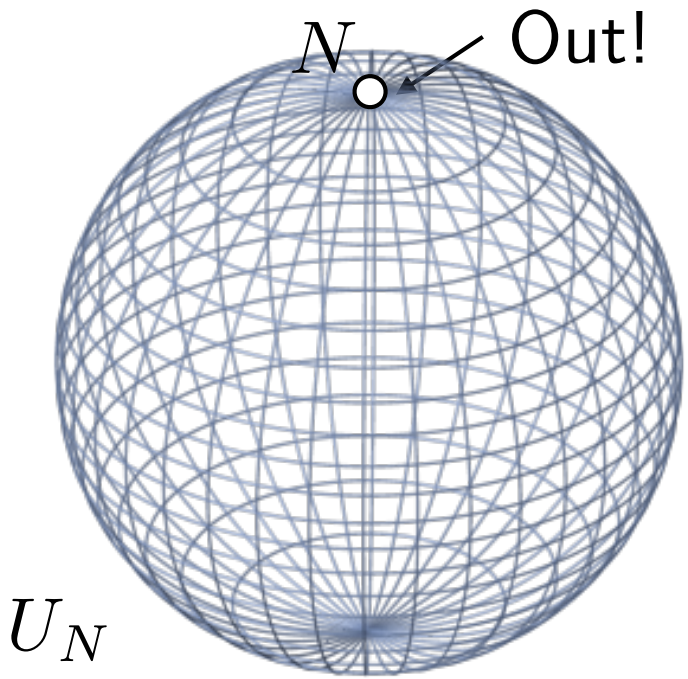
$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$



Examples

- Consider the open cover consisting of

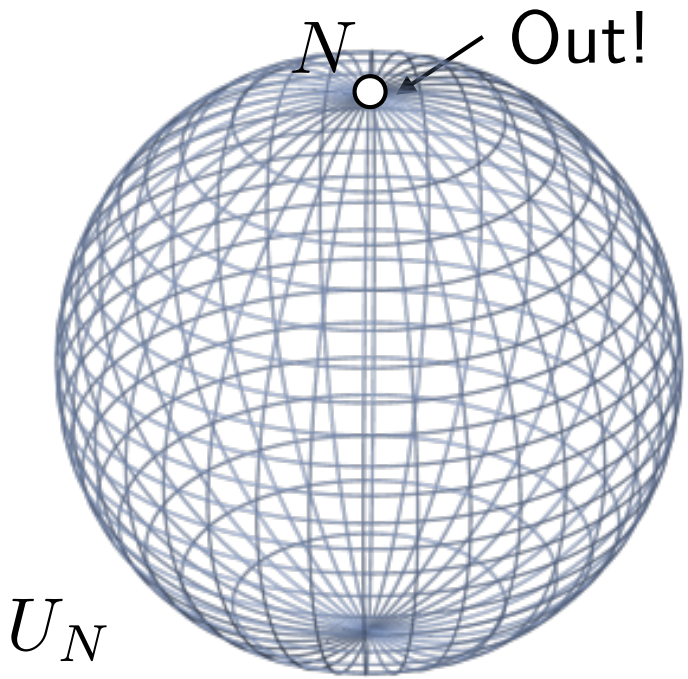
$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$

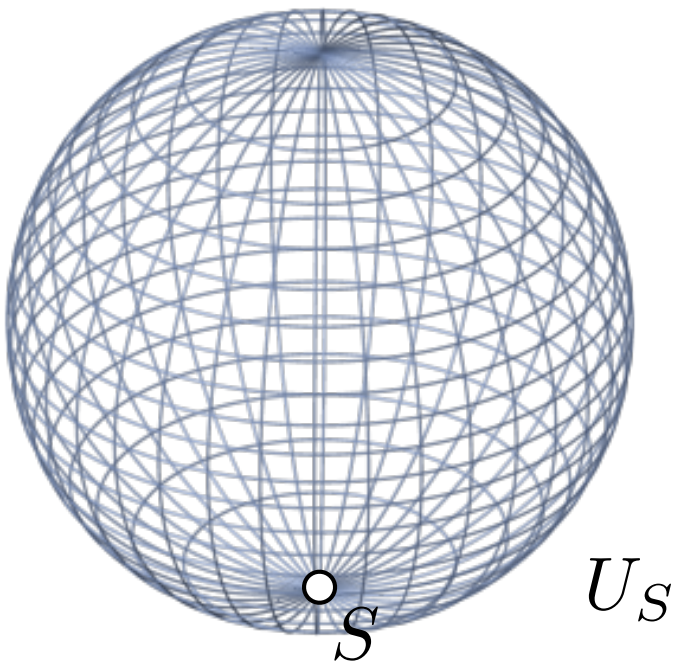


Examples

- Consider the open cover consisting of

$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$

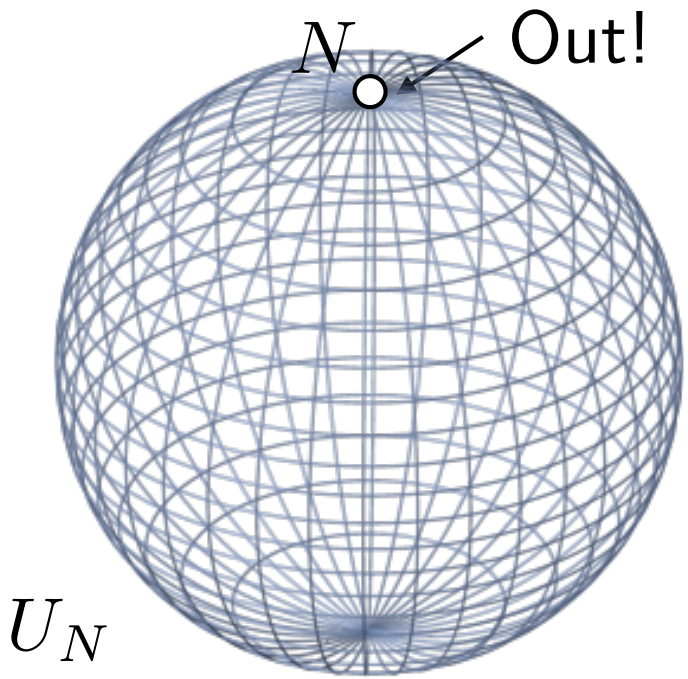


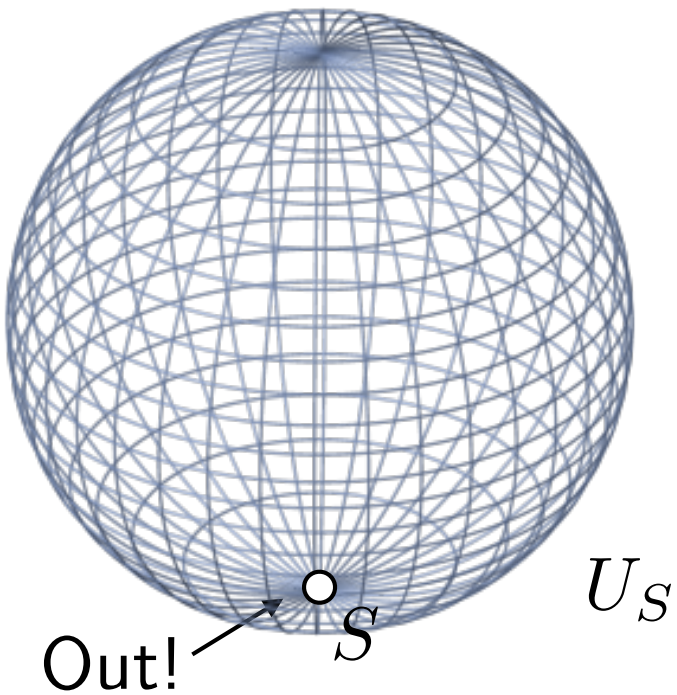


Examples

- Consider the open cover consisting of

$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$





Examples

Examples

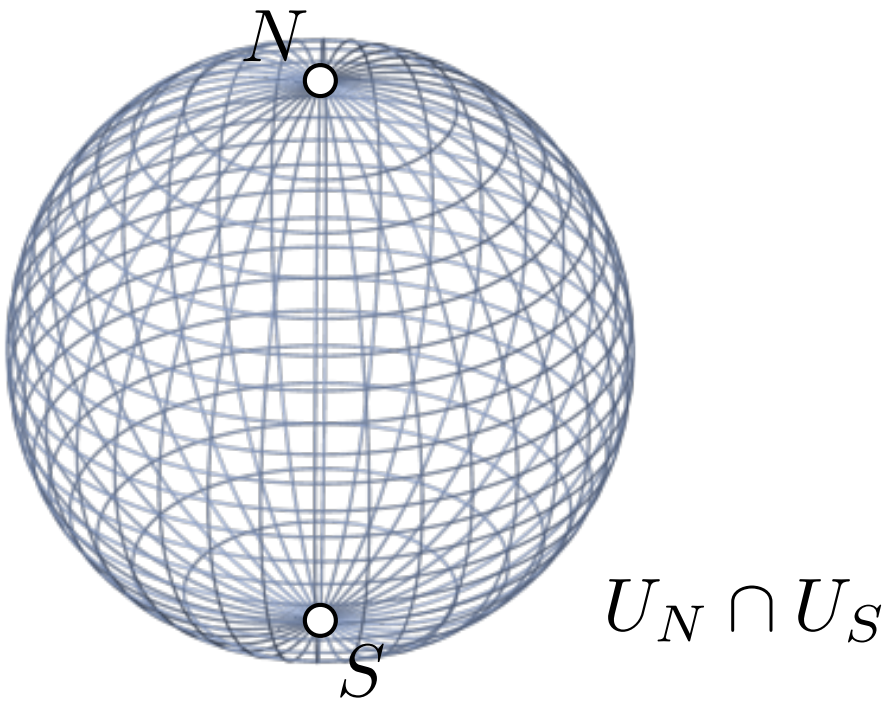
- On the overlap,

$$U_N \cap U_S = S^n - \{N, S\}\,.$$

Examples

- On the overlap,

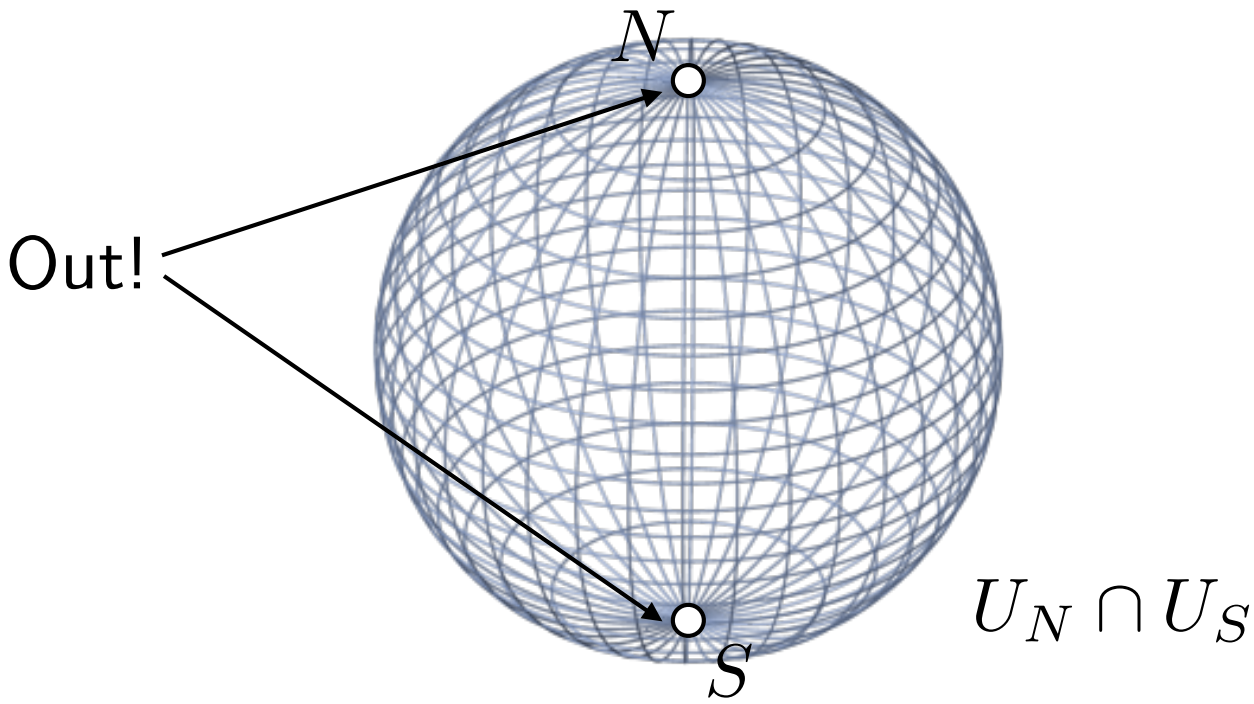
$$U_N \cap U_S = S^n - \{N, S\}\,.$$



Examples

- On the overlap,

$$U_N \cap U_S = S^n - \{N, S\}\,.$$



Examples

Examples

- The transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n)\,.$$

Examples

Examples

- Consequently,

$$(U_N, \sigma_N) \quad \text{and} \quad (U_S, \sigma_S)$$

form a smooth atlas for S^n .

Examples

- Consequently,

$$(U_N, \sigma_N) \quad \text{and} \quad (U_S, \sigma_S)$$

form a smooth atlas for S^n .

- So, the sphere is a smooth manifold.

Conclusions

Conclusions

- In lecture 3 of this mini-course we will show that a manifold can be **reconstructed** from its transition functions.

Conclusions

- In lecture 3 of this mini-course we will show that a manifold can be **reconstructed** from its transition functions.

- Such a construction was first proposed by Andre Weil around 1944 in his book, *Foundations of Algebraic Geometry*.

Conclusions

- In lecture 3 of this mini-course we will show that a manifold can be **reconstructed** from its transition functions.

- Such a construction was first proposed by Andre Weil around 1944 in his book, *Foundations of Algebraic Geometry*.

- A similar approach was used to construct fiber bundles in the 1950's (Steenrod).