

Parametric Pseudo-Manifolds and Applications in Graphics

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February 2-6

Schedule

February 2, 2009 (Monday), 1-3 PM

1. Overview of Manifolds
2. Applications in Graphics and Vision

February 3, 2009 (Tuesday), 1-3 PM

3. Constructing Manifolds
4. An Application: Surface Fitting

February 5, 2009 (Thursday), 1-3 PM

5. A Manifold-Based Construction for Surface Fitting

February 6, 2009 (Friday), 1-3 PM

6. Implementation Details

Overview of Manifolds

Lecture 1 - February 2, 2009 - 1-2 PM

Outline

- Manifolds: Brief History
- Informal definition
- Formal definition
- Example
 - The Sphere
- Conclusions

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Origins of Manifolds

- Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.
- In a famous paper published in 1888, Dyck already uses the term **manifold** (in German).
- In the early 1900's, Dehn, Heegaard, Veblen and Alexander routinely used the term **manifold**.
- Hermann Weyl was among the first to give a rigorous definition (1913).

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Keys Contributors to the notion of manifold:

Georg Friedrich Bernhard Riemann
1826-1866



Hermann Klaus Hugo Weyl
1885-1955



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Keys Contributors to the notion of manifold:

Hermann Weyl (again)

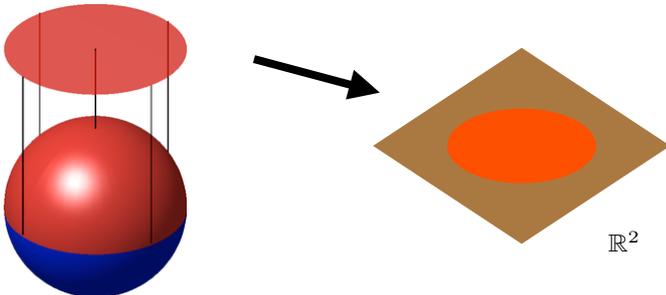


Hassler Whitney
1907-1989



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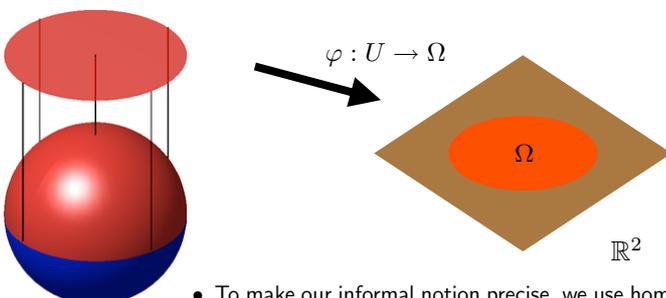
Why we need Manifolds?



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Manifolds: Informal Definition

- A manifold is a topological space with an open cover so that every open set in this cover "looks" like an open subset of \mathbb{R}^n .

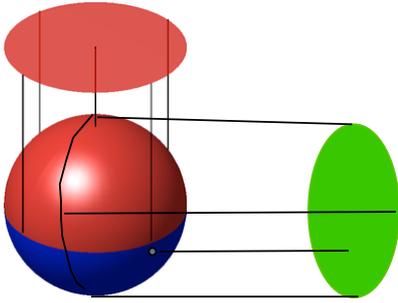


- To make our informal notion precise, we use homeomorphisms, $\varphi : U \rightarrow \Omega$, where $\Omega \subseteq \mathbb{R}^n$ is an open subset of \mathbb{R}^n .

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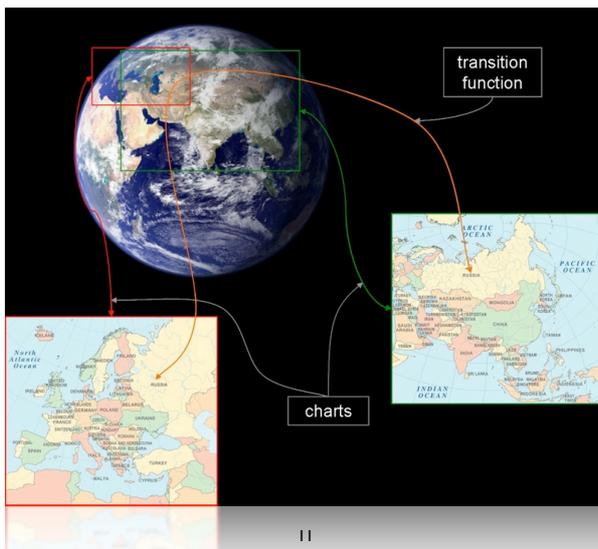
Manifolds: Informal Definition

- We also want to be able “to do calculus” on our manifolds. For this we need some conditions on **overlaps** of open sets.



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Manifold: An Intuitive Picture

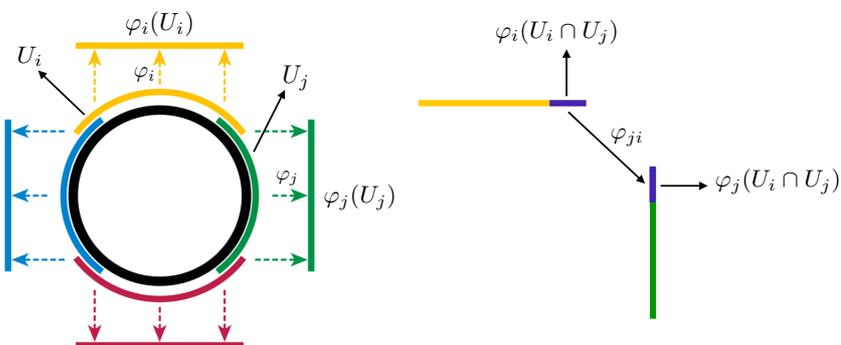


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Manifolds: Informal Definition

- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

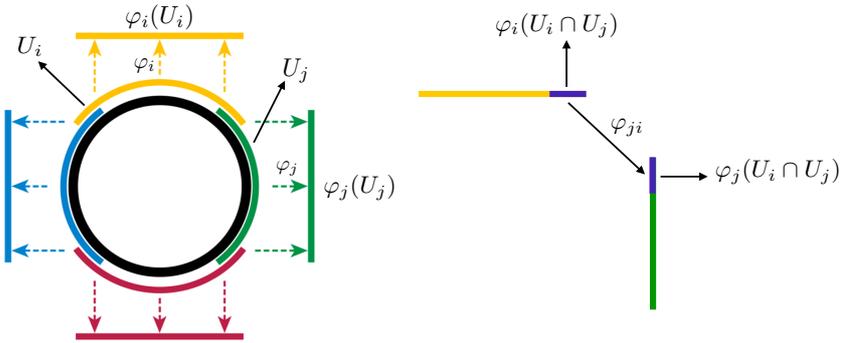
$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j).$$



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Manifolds: Informal Definition

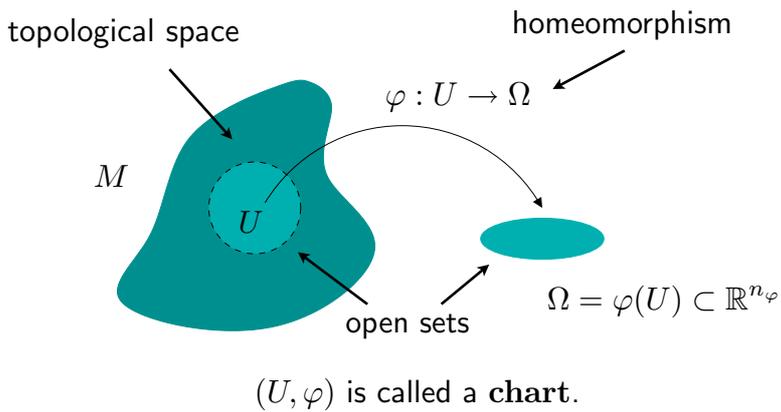
- This is a map between two open subsets of \mathbb{R}^n and we require it possess a certain amount of **smoothness**.



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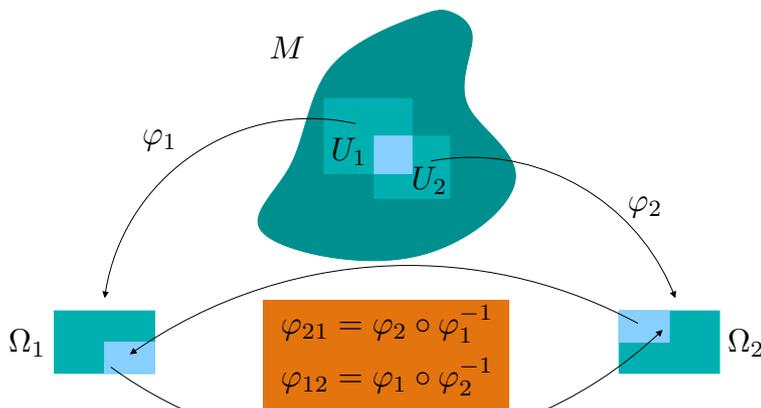
Manifolds: Formal Definition

Recall the definition of a manifold...



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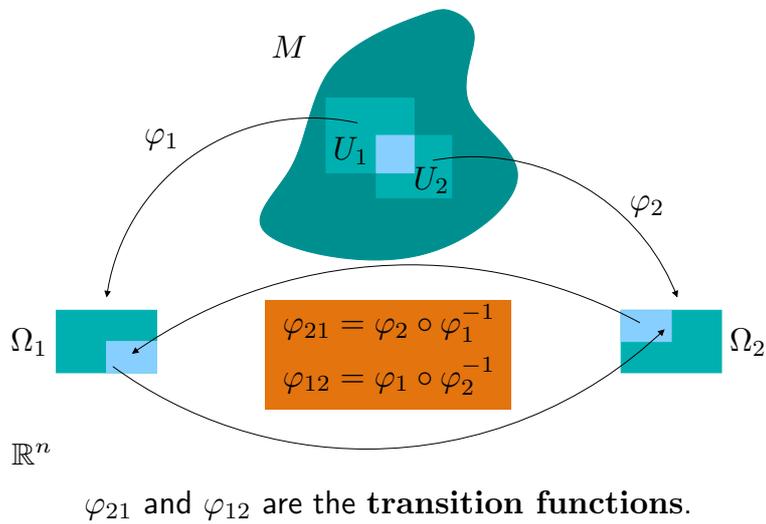
Manifolds: Formal Definition



$$\begin{aligned} \varphi_{21} &: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2) \\ \varphi_{12} &: \varphi_2(U_1 \cap U_2) \rightarrow \varphi_1(U_1 \cap U_2) \end{aligned}$$

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Manifolds: Formal Definition



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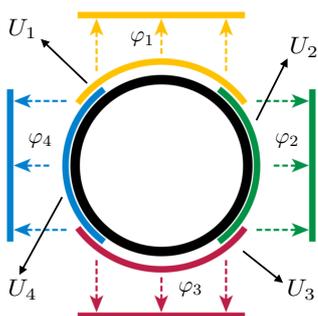
Manifolds: Formal Definition

A C^k n -atlas is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

- (1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i .
- (2) $M = \bigcup_{i \in I} U_i$.
- (3) Whenever $U_i \cap U_j \neq \emptyset$, the transition function φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism.

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Manifolds: Formal Definition



$$M = \bigcup_{i=1}^4 U_i$$

φ_i is a C^k diffeomorphism

Atlas: $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$

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Manifolds: Formal Definition

The existence of a C^k atlas on a topological space, M , is sufficient to establish that M is an n -dimensional C^k manifold, but...

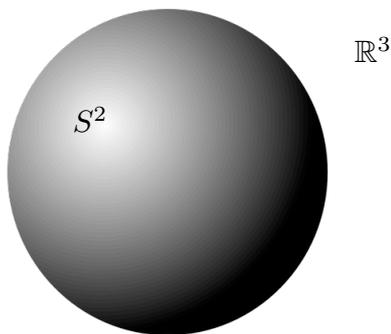
- there may be many choice of atlases;
- we get around this problem by defining a notion of atlas compatibility;
- this notion induces an equivalence relation of atlases on M ;
- the set of all charts compatible with a given atlas is a maximum atlas in its class.

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Examples

- The sphere

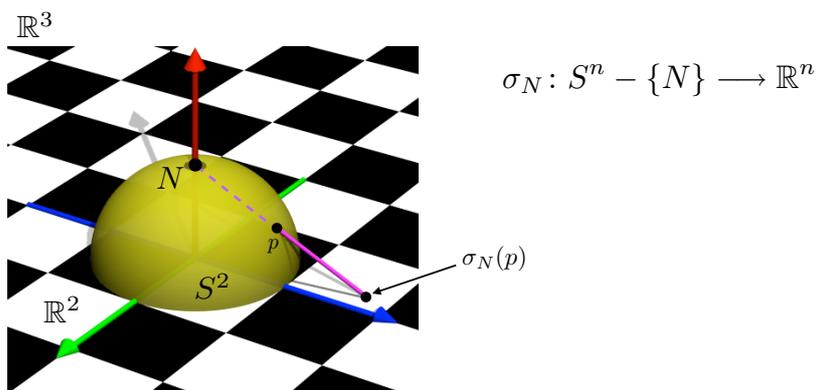
$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}.$$



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Examples

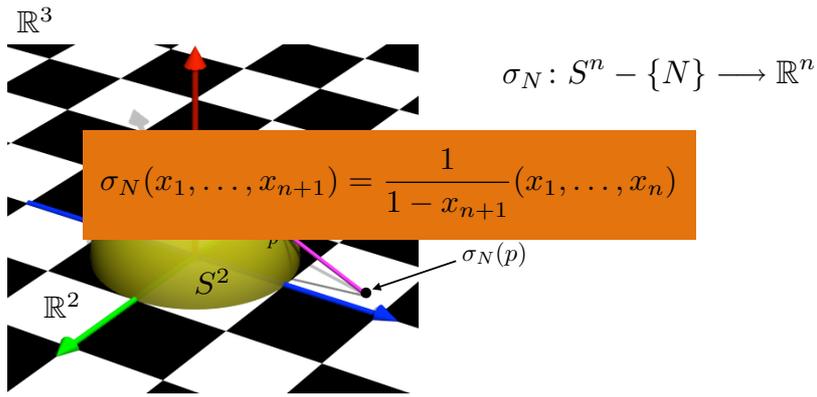
- We use stereographic projection from the north pole ...



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Examples

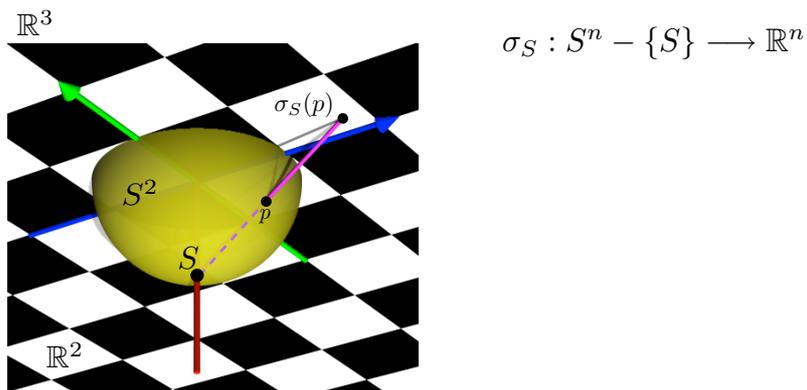
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Examples

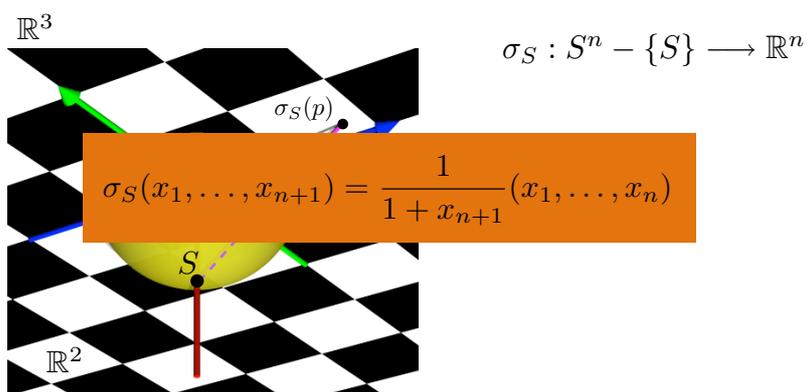
and from the south pole:



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Examples

and from the south pole:



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Examples

- Inverse stereographic projections:

$$\sigma_N^{-1}(x_1, \dots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \dots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

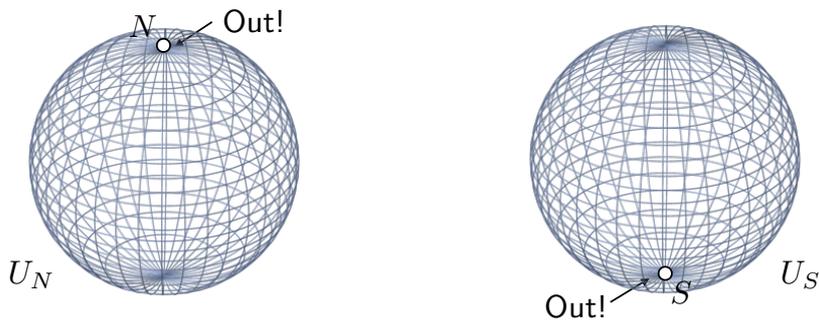
$$\sigma_S^{-1}(x_1, \dots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \dots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

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Examples

- Consider the open cover consisting of

$$U_N = S^n - \{N\} \quad \text{and} \quad U_S = S^n - \{S\}.$$

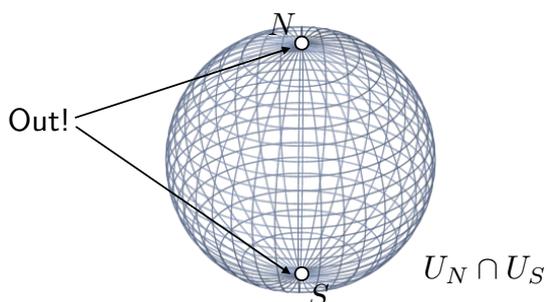


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Examples

- On the overlap,

$$U_N \cap U_S = S^n - \{N, S\}.$$



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Examples

- The transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \dots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2} (x_1, \dots, x_n).$$

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Examples

- Consequently,

$$(U_N, \sigma_N) \quad \text{and} \quad (U_S, \sigma_S)$$

form a smooth atlas for S^n .

- So, the sphere is a smooth manifold.

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Conclusions

- In lecture 3 of this mini-course we will show that a manifold can be **reconstructed** from its transition functions.
- Such a construction was first proposed by Andre Weil around 1944 in his book, *Foundations of Algebraic Geometry*.
- A similar approach was used to construct fiber bundles in the 1950's (Steenrod).

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