

# **Constructing Manifolds**

Lecture 3 - February 3, 2009 - 1-2 PM

# Outline

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- Sets of gluing data
- The cocycle condition
- Parametric pseudo-manifolds (PPM's)
- Conclusions

# Sets of Gluing Data

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Let  $n$  and  $k$  be integers such that  $n \geq 1$  and  $k \geq 1$  (or  $k = \infty$ ).

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A set of gluing data is a triple

$$\mathcal{G} = \left( (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K} \right),$$

where  $I$  and  $K$  are countable sets and  $I$  is non-empty, satisfying the following three properties:

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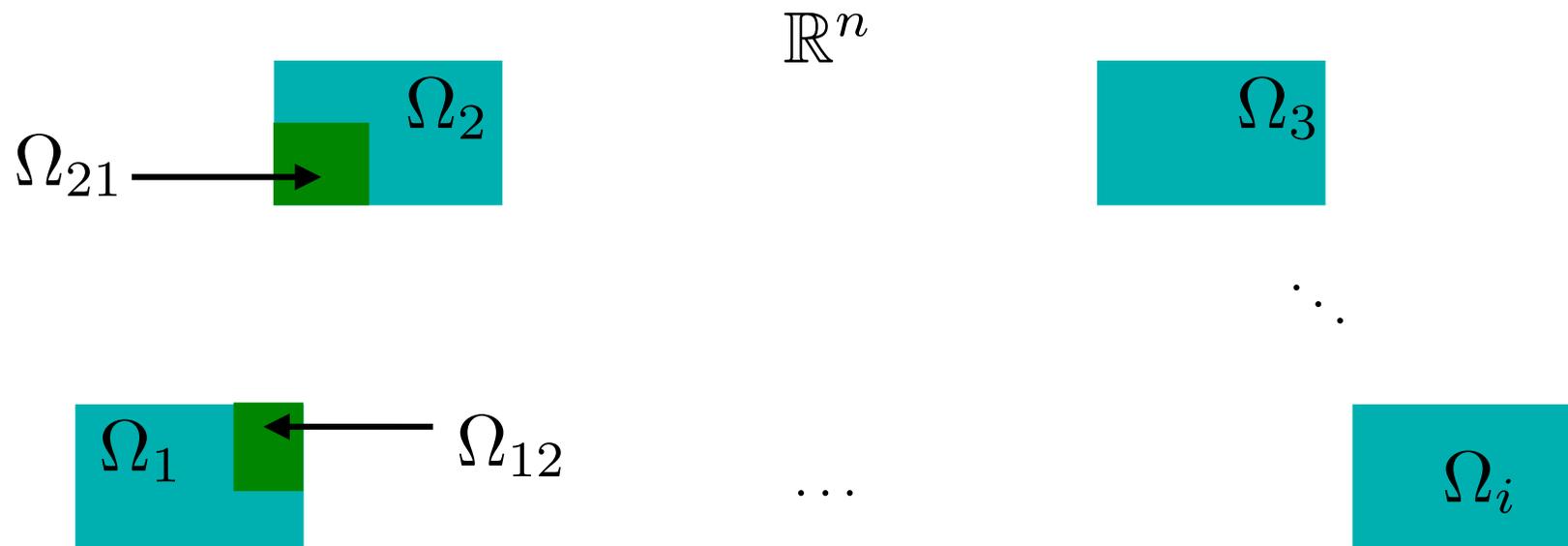
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(2) For every pair  $(i, j) \in I \times I$ , the set  $\Omega_{ij}$  is an open subset of  $\Omega_i$ . Furthermore,  $\Omega_{ii} = \Omega_i$ , and  $\Omega_{ji} \neq \emptyset$  if and only if  $\Omega_{ij} \neq \emptyset$ . Each non-empty  $\Omega_{ij}$  (with  $i \neq j$ ) is called **gluing domain**.



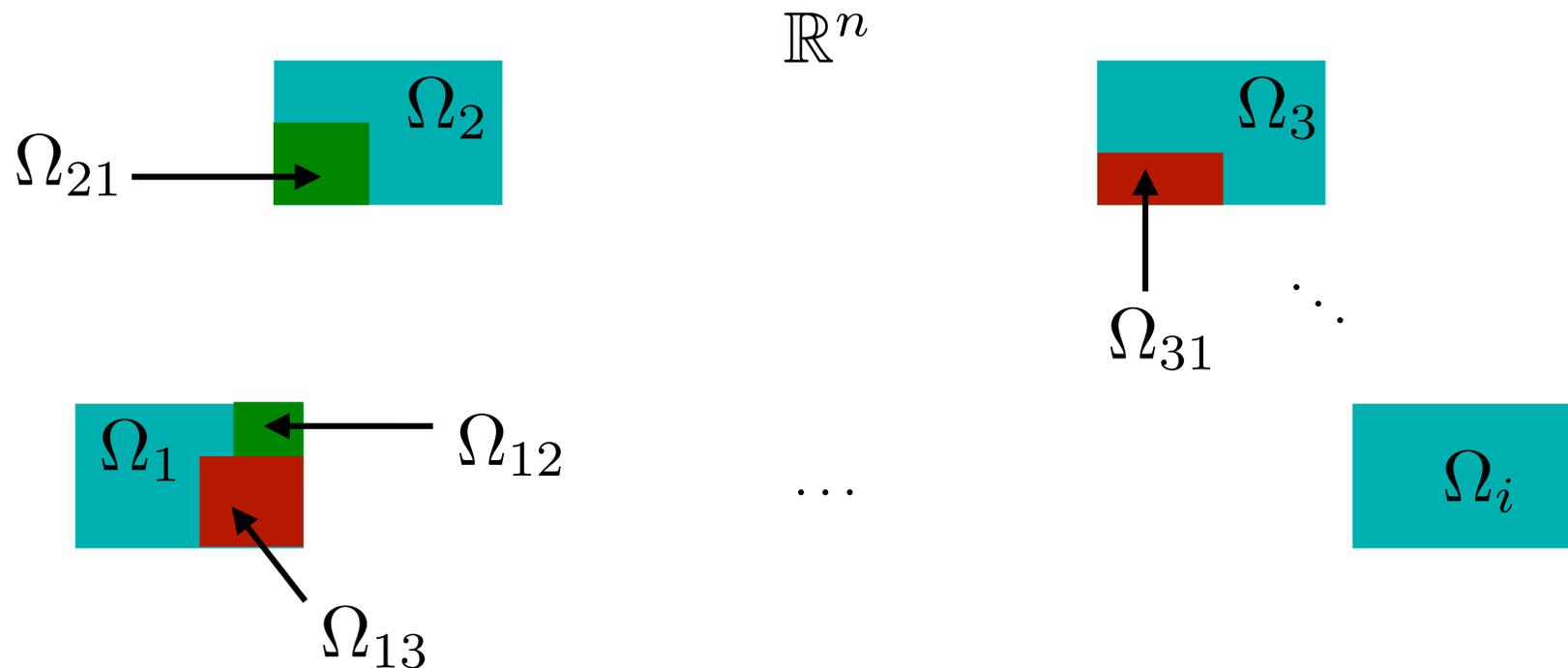
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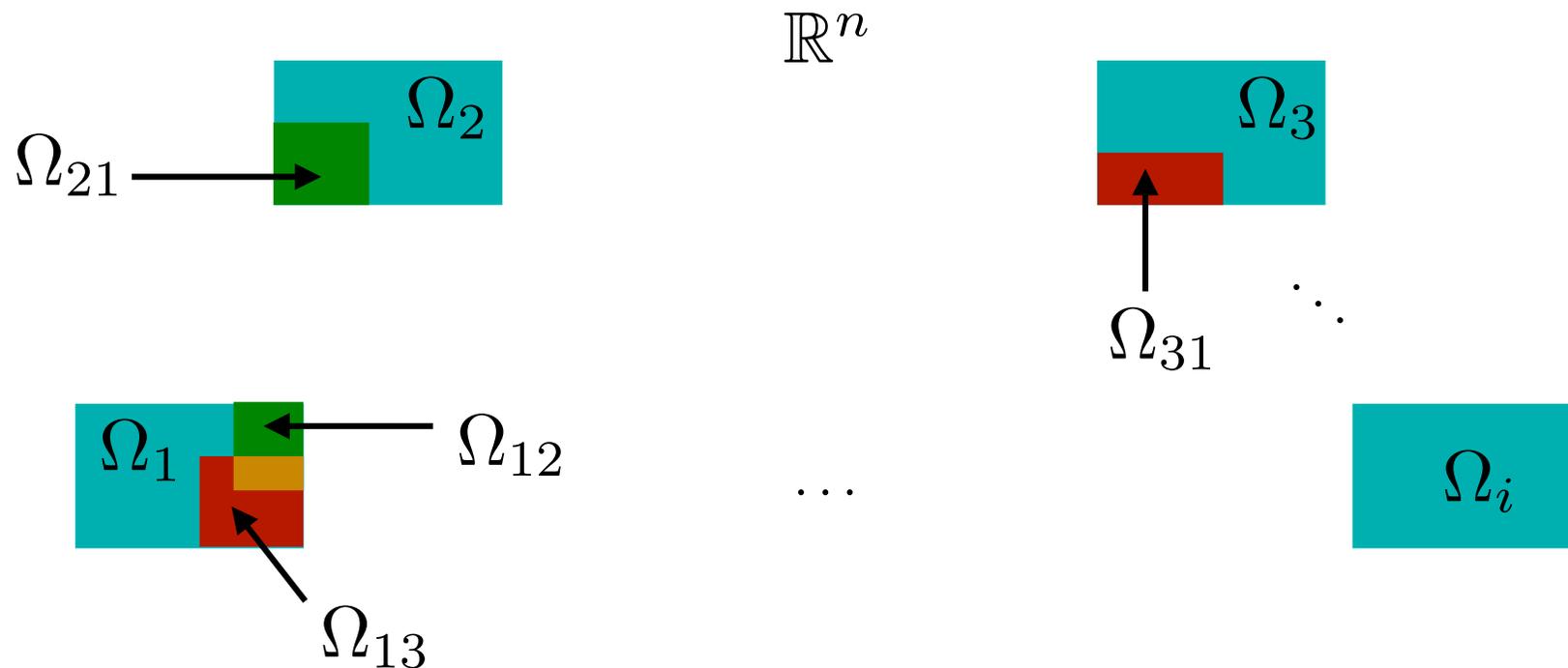
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(3) If we let

$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then

$$\varphi_{ji} : \Omega_{ij} \longrightarrow \Omega_{ji}$$

is a  $C^k$  bijection for every  $(i, j) \in K$ , called a **transition function** or **gluing function**.

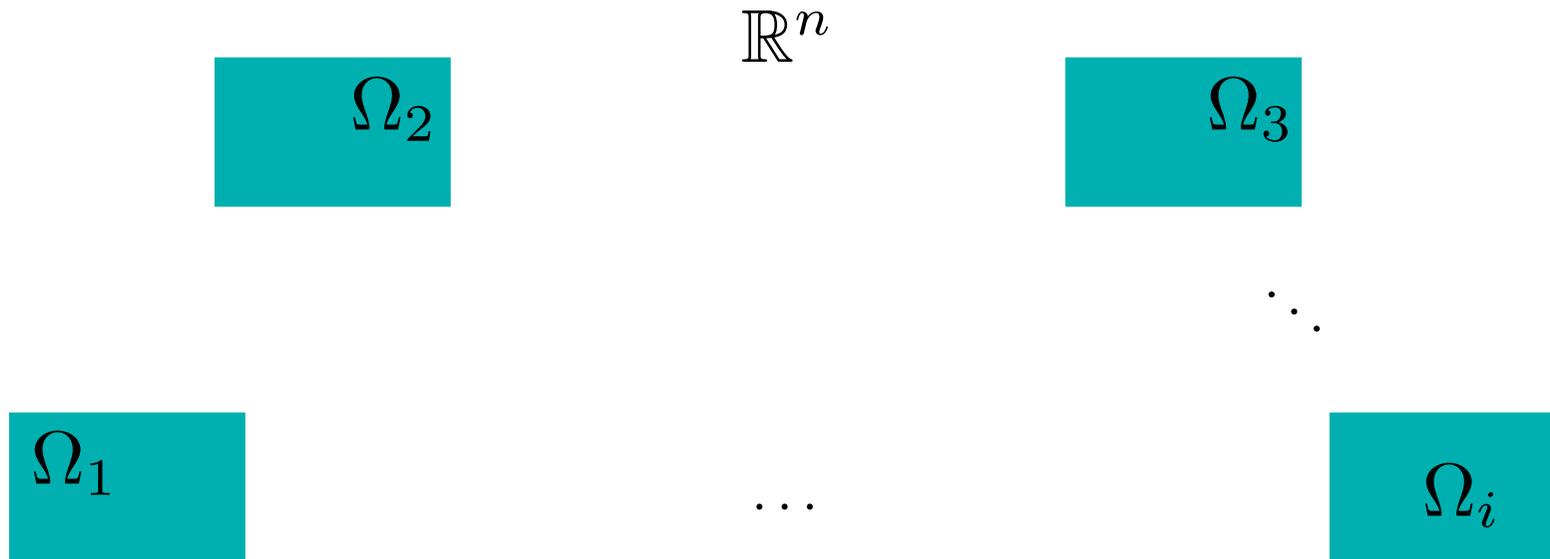
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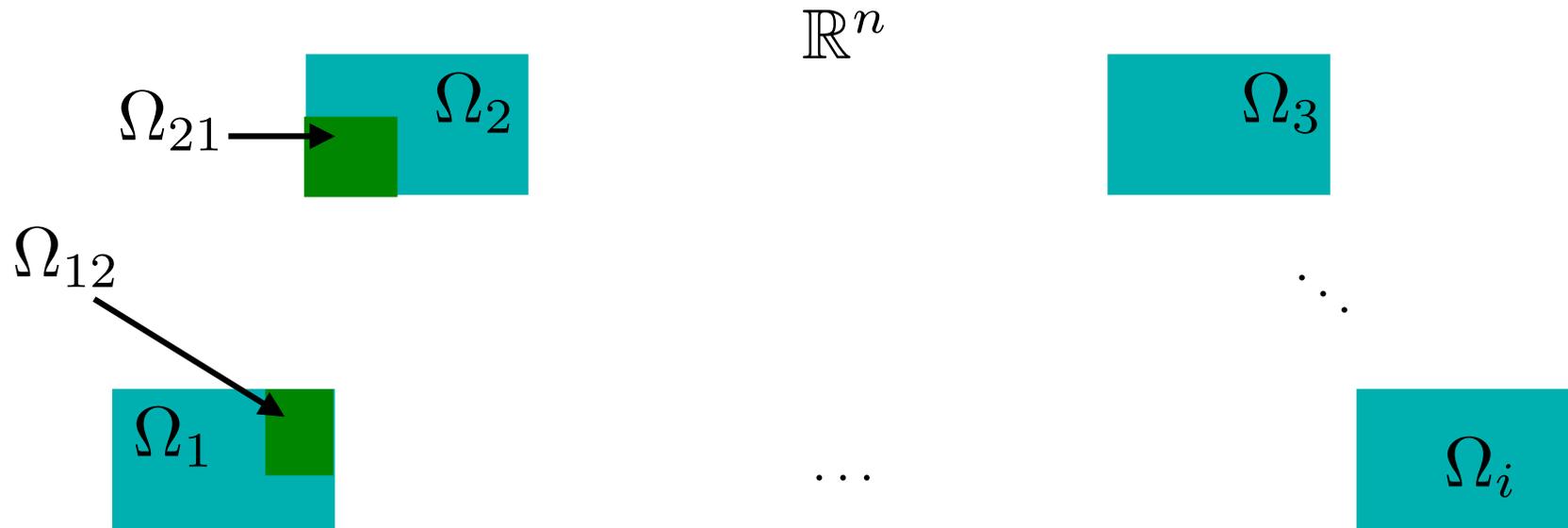
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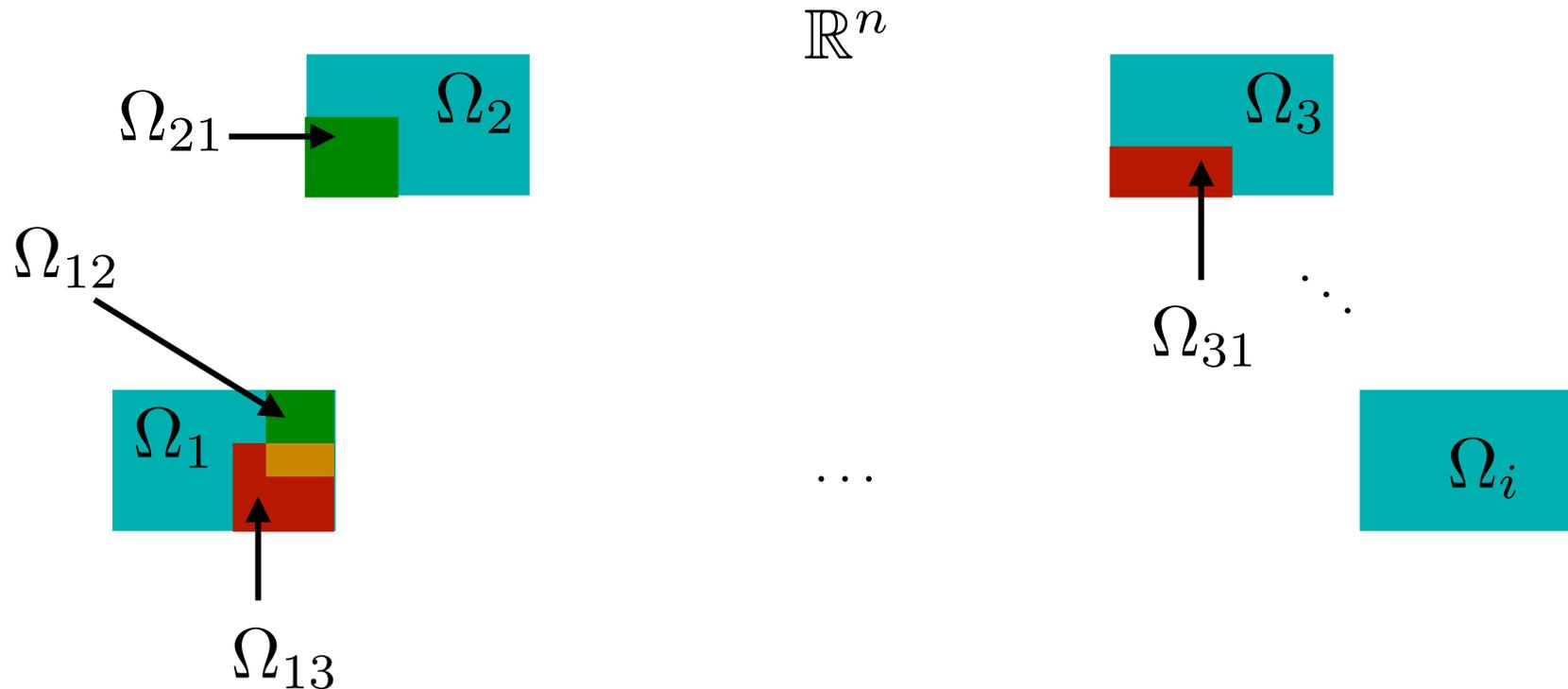
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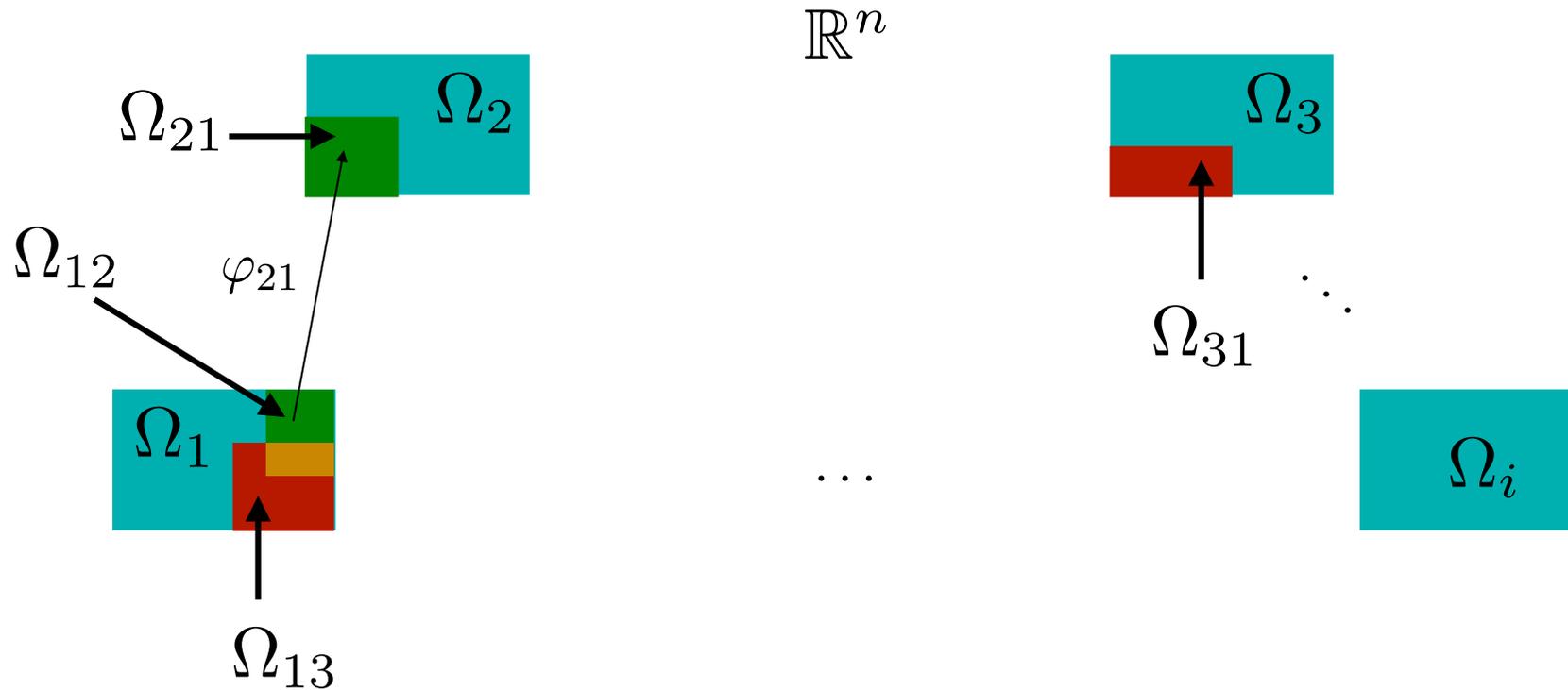
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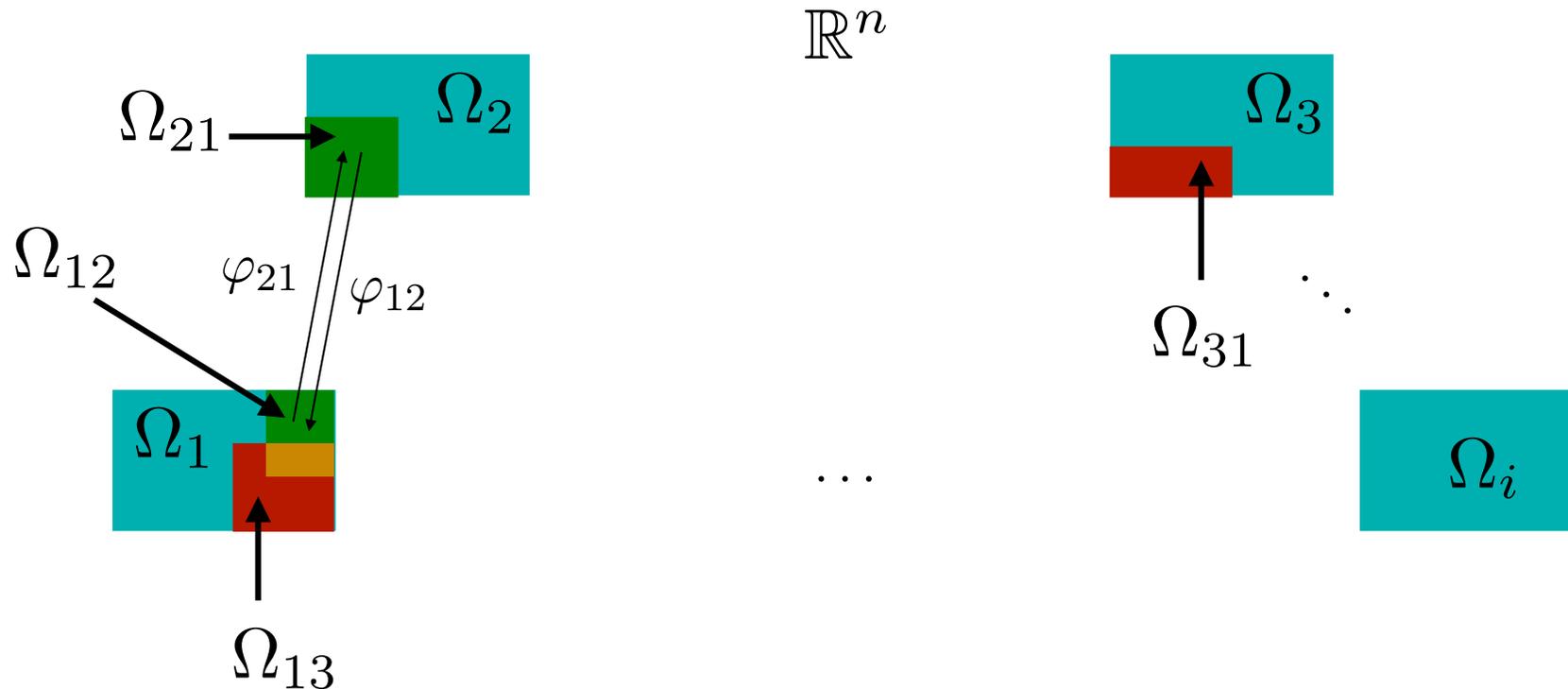
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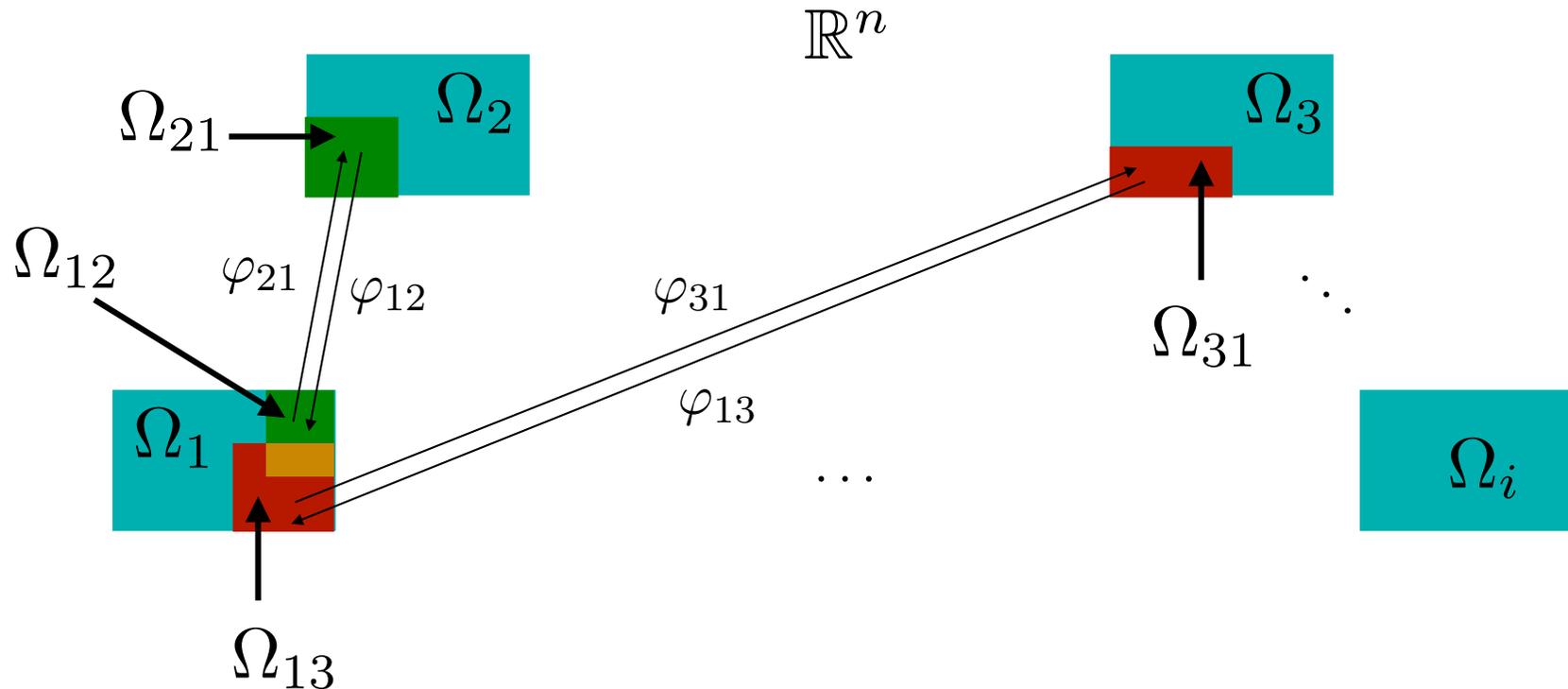
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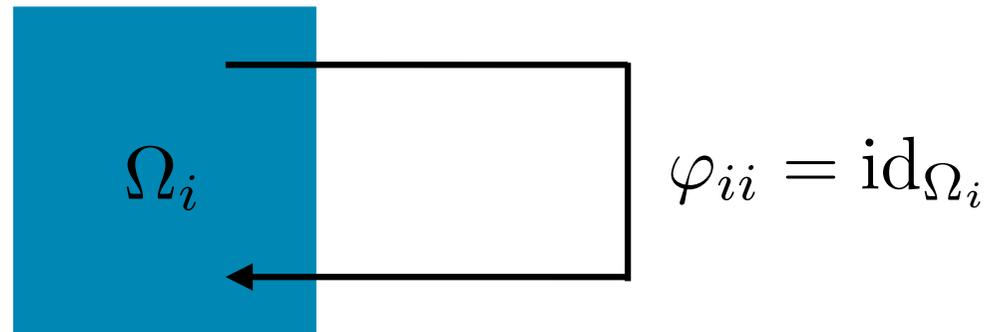
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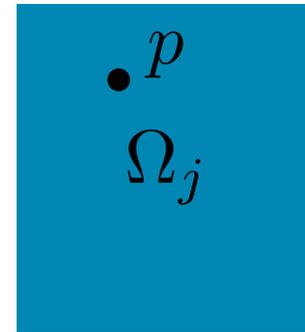
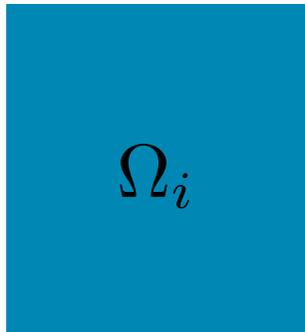
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(b)  $\varphi_{ij} = \varphi_{ji}^{-1}$ , for all  $(i, j) \in K$ , and

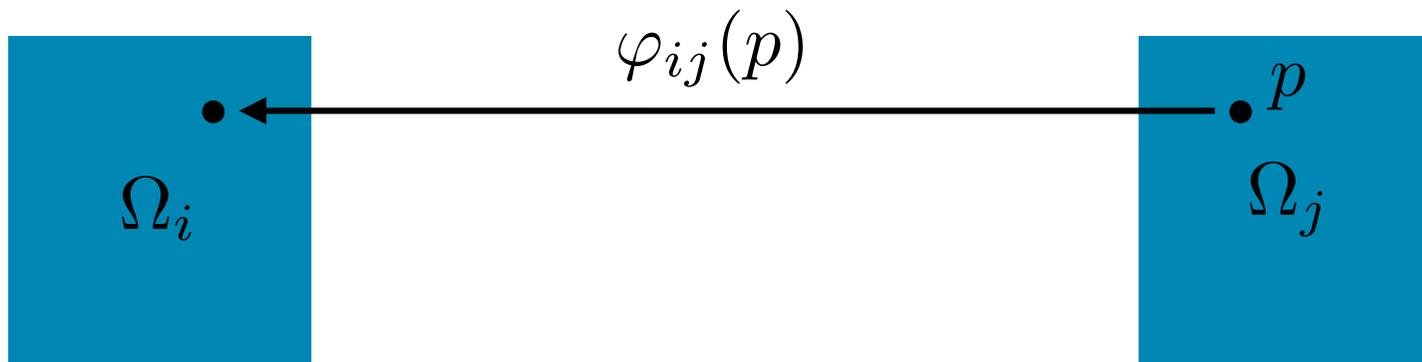
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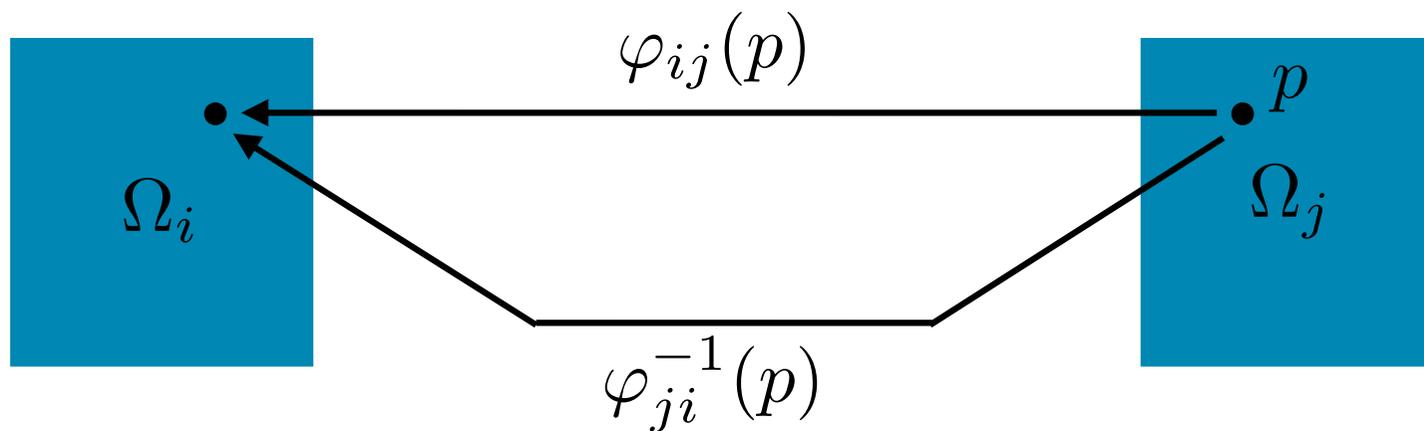
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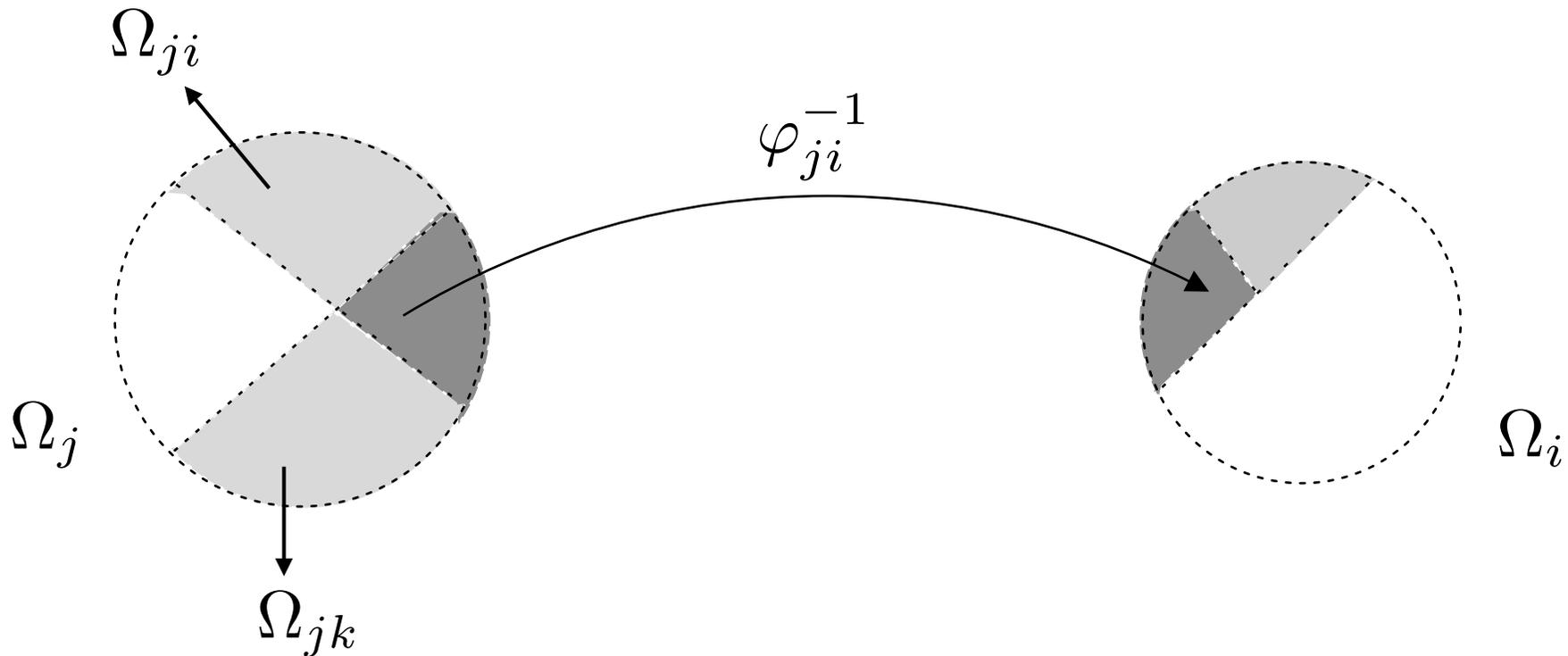
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(c) for all  $i, j$ , and  $k$ , if  $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$  then  $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$  and  $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$ , for all  $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$ .

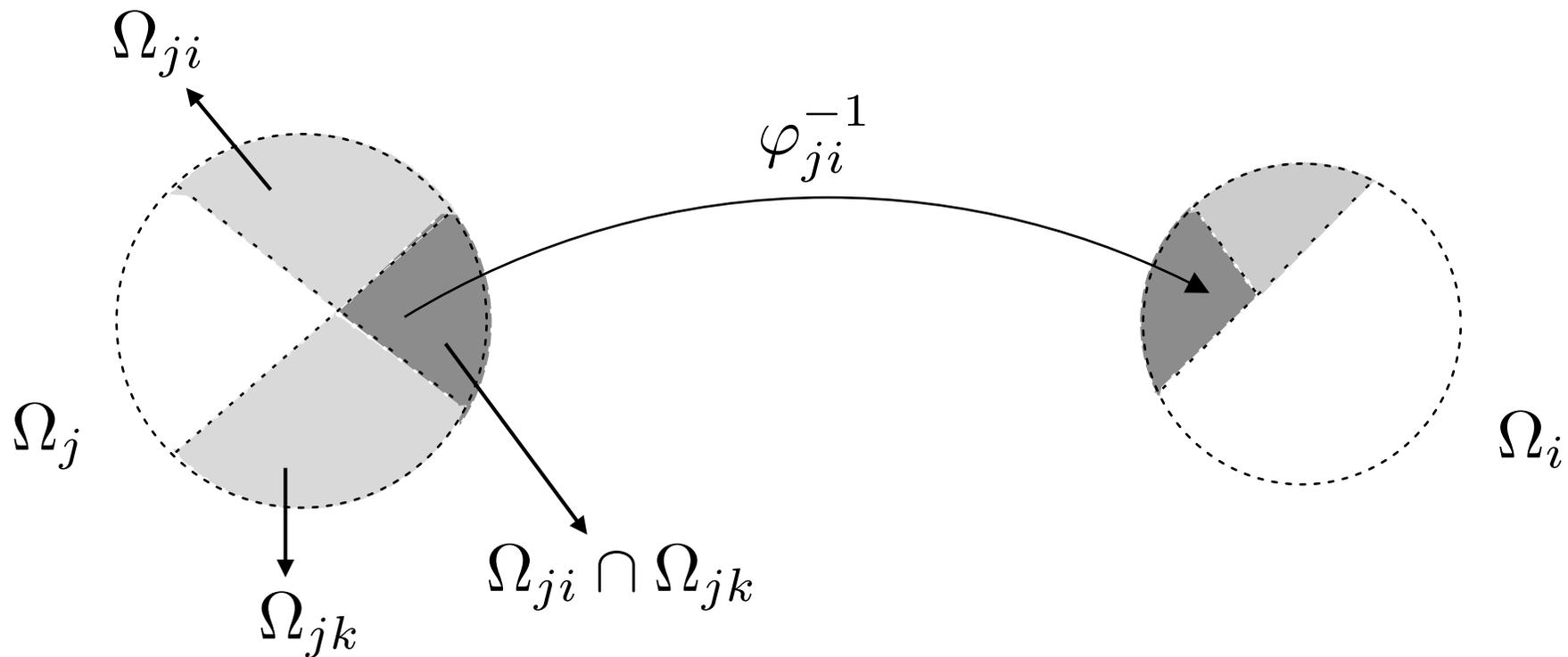
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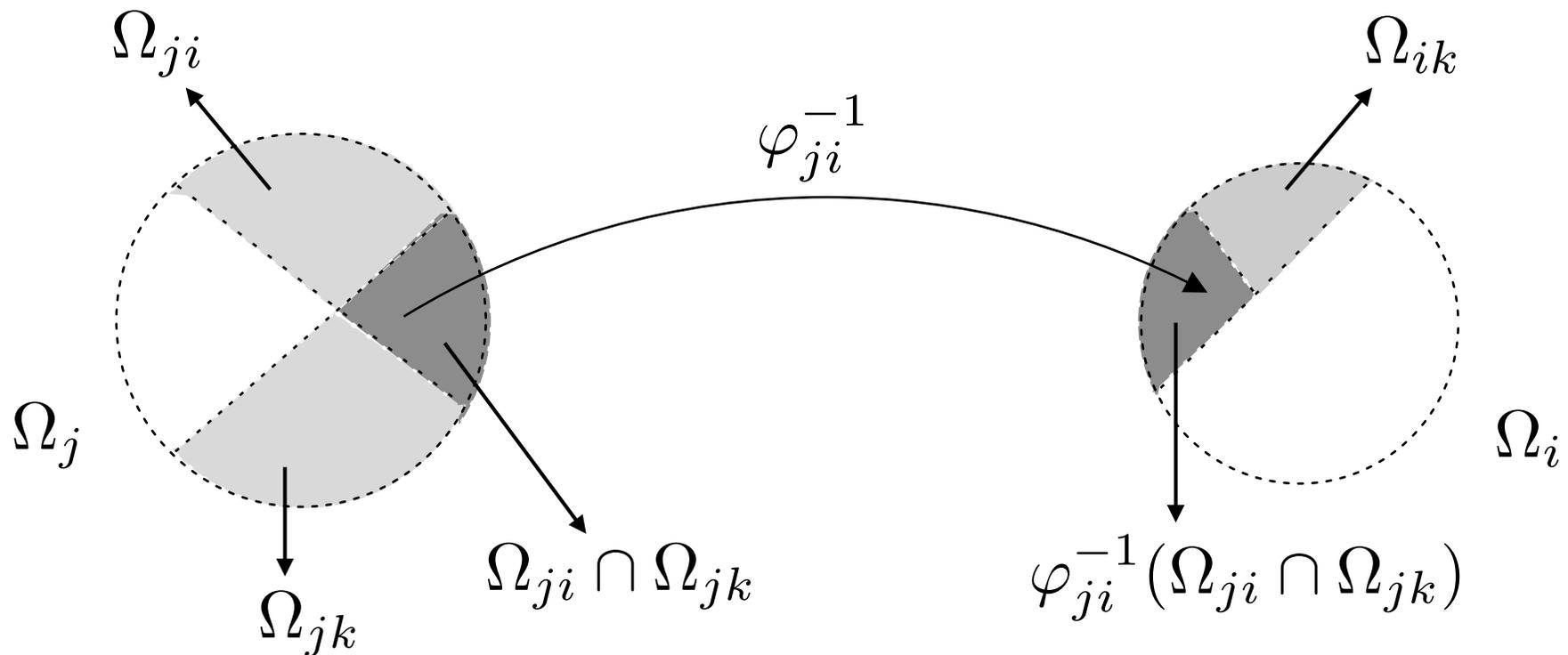
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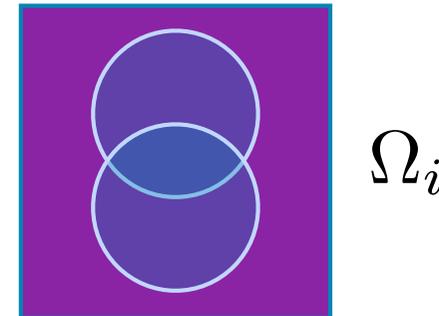
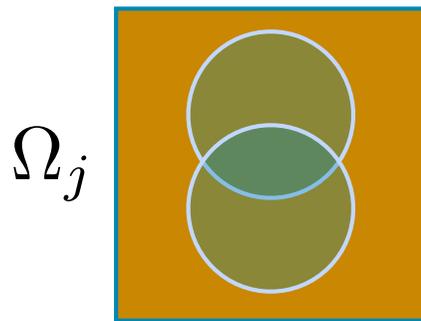
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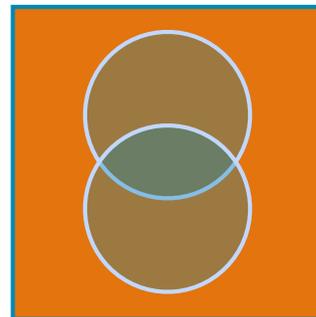
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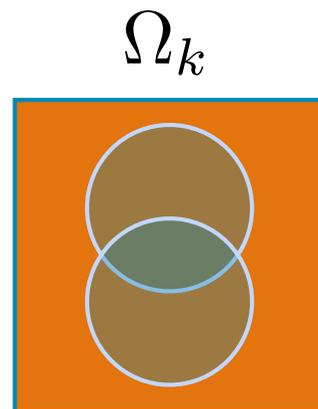
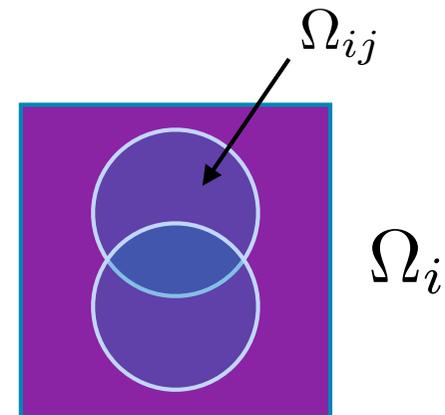
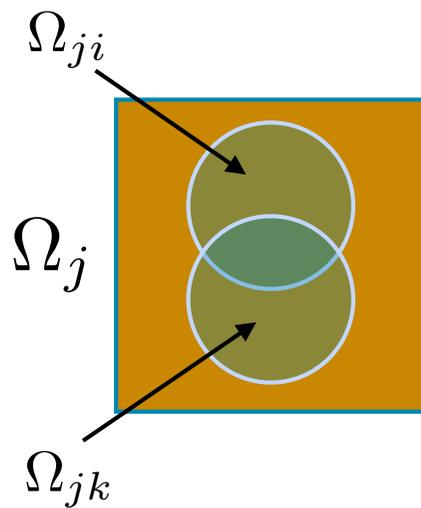


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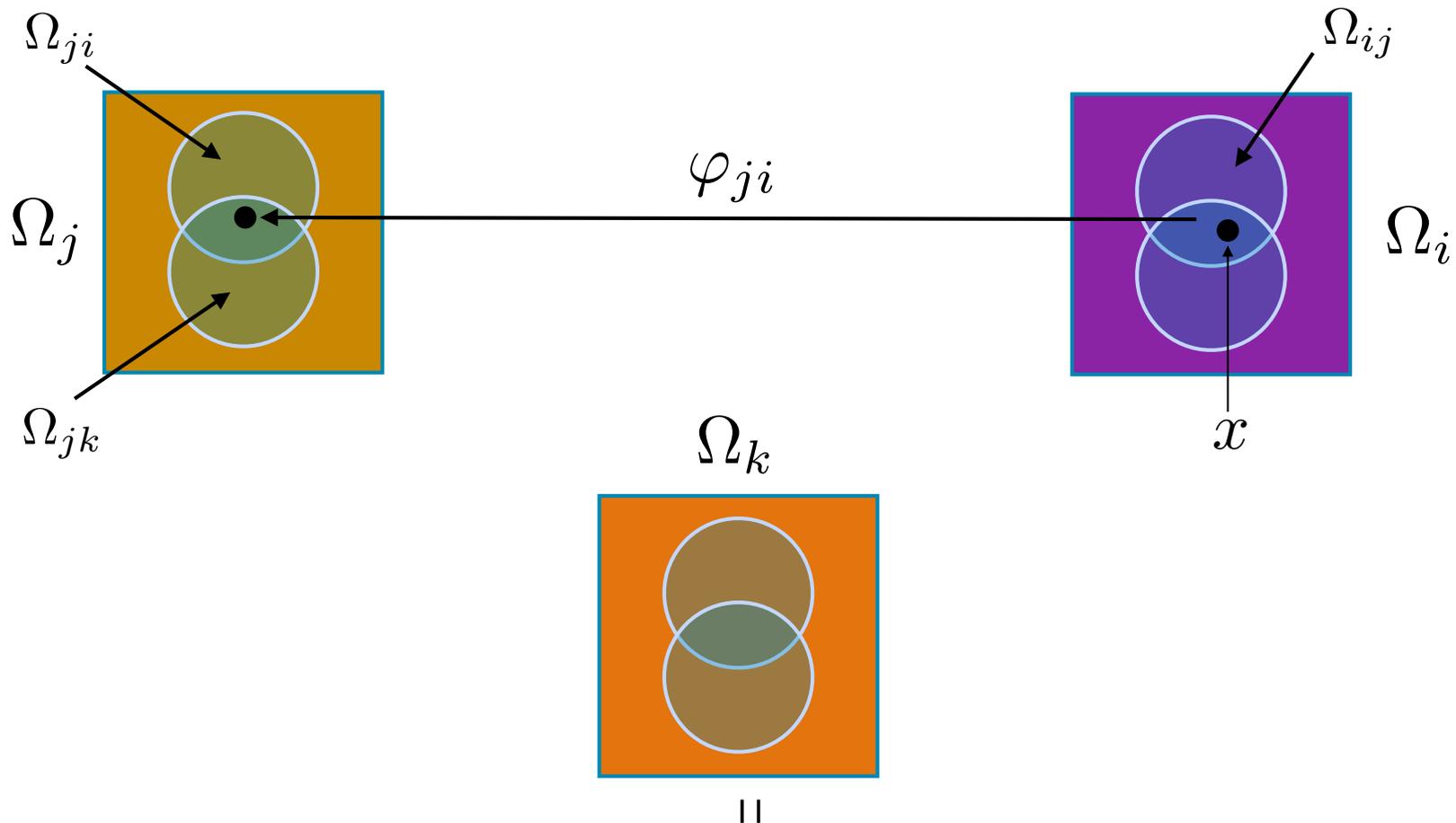
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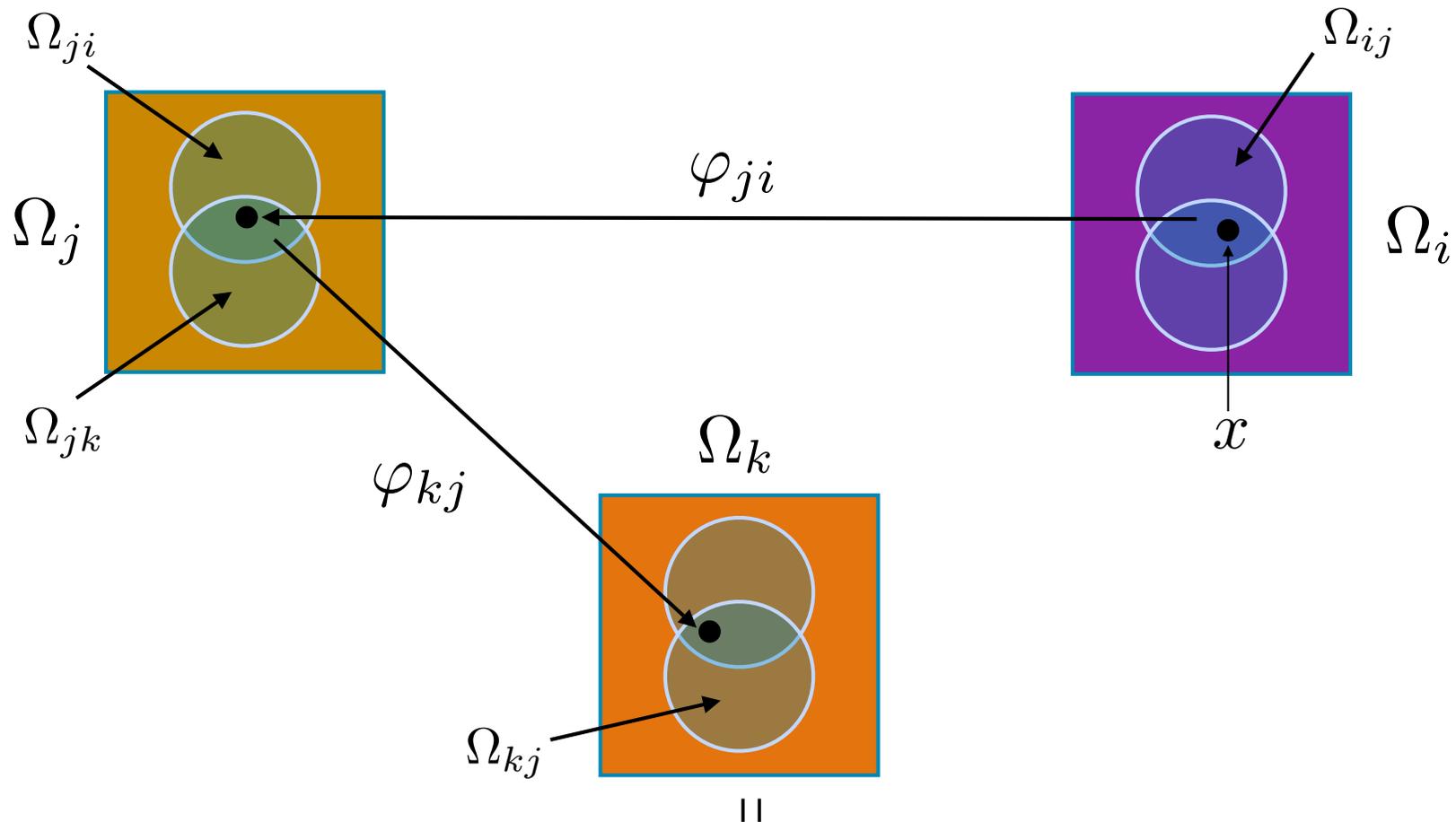
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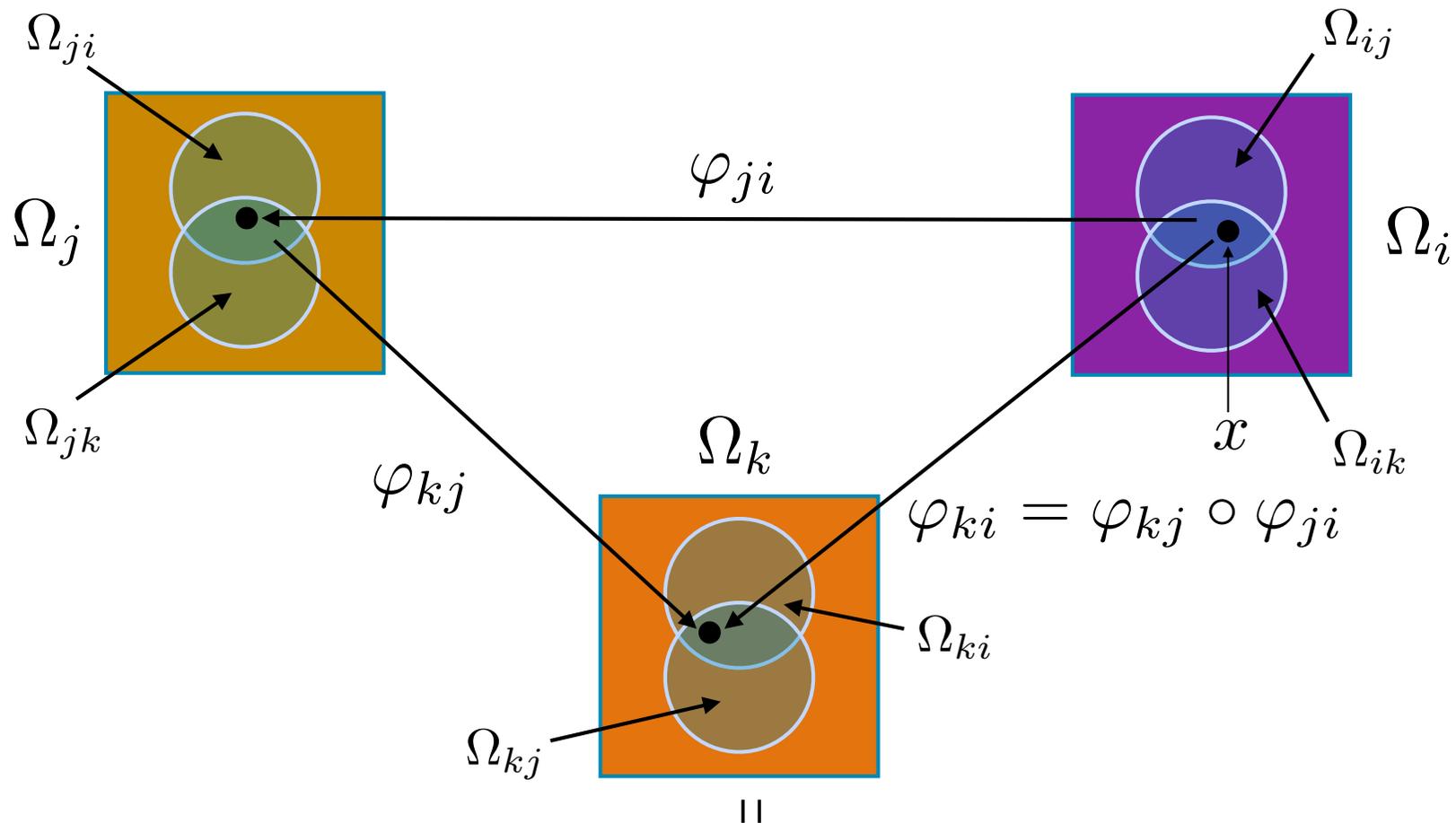
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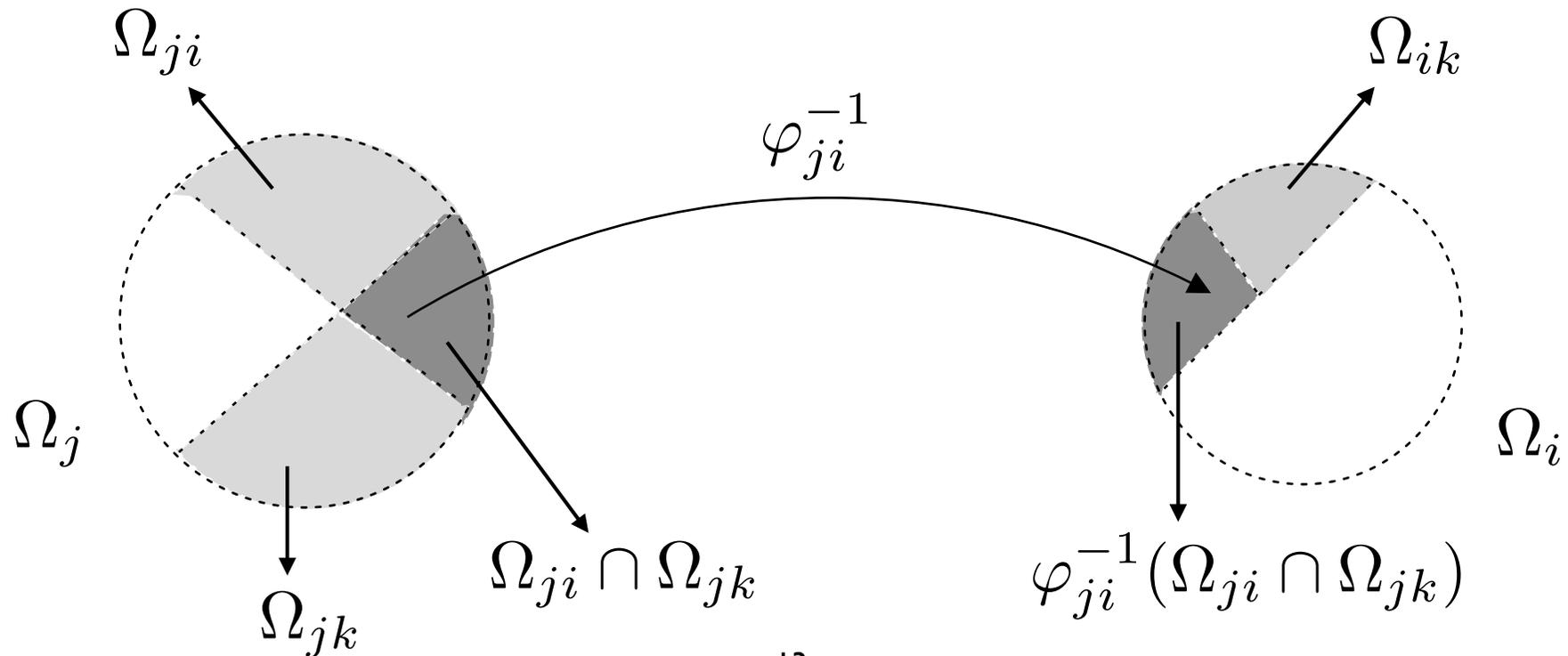
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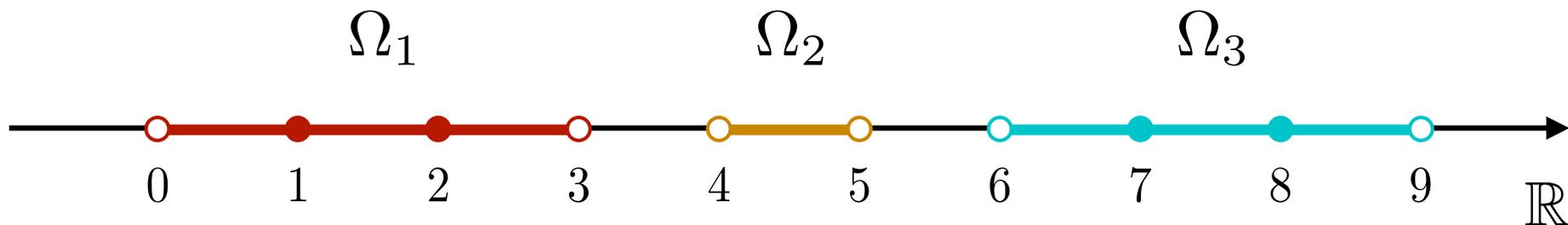
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$$\Omega_{12} = (0, 1) \quad \text{and} \quad \Omega_{13} = (2, 3),$$

$$\Omega_{21} = \Omega_{23} = (4, 5), \quad \text{and}$$

$$\Omega_{32} = (8, 9) \quad \text{and} \quad \Omega_{31} = (6, 7).$$

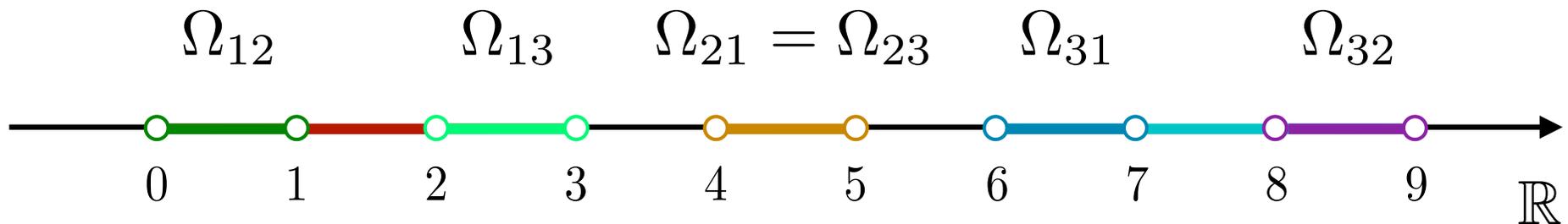
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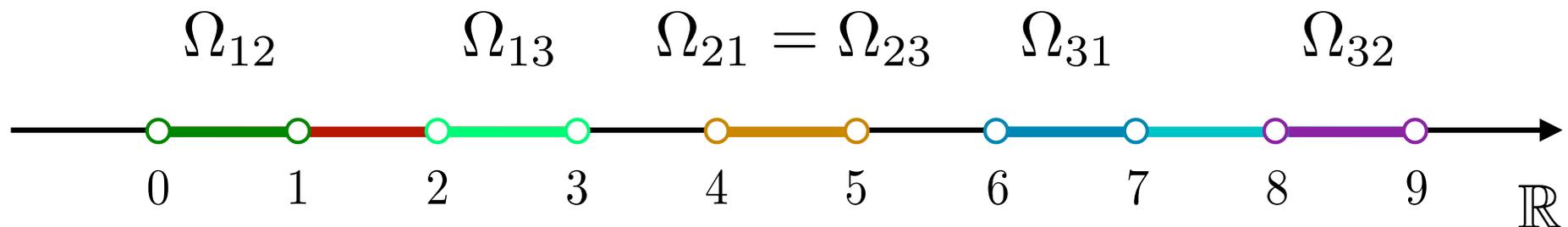
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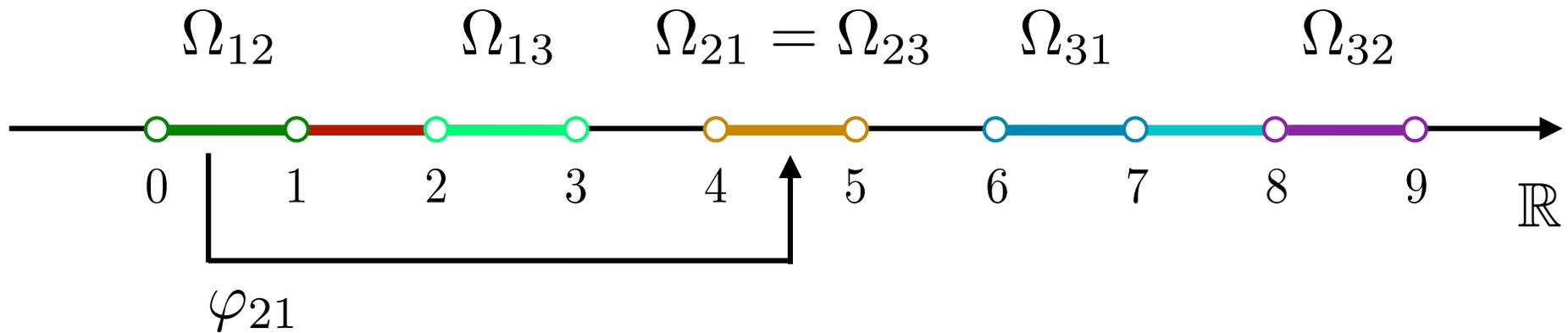
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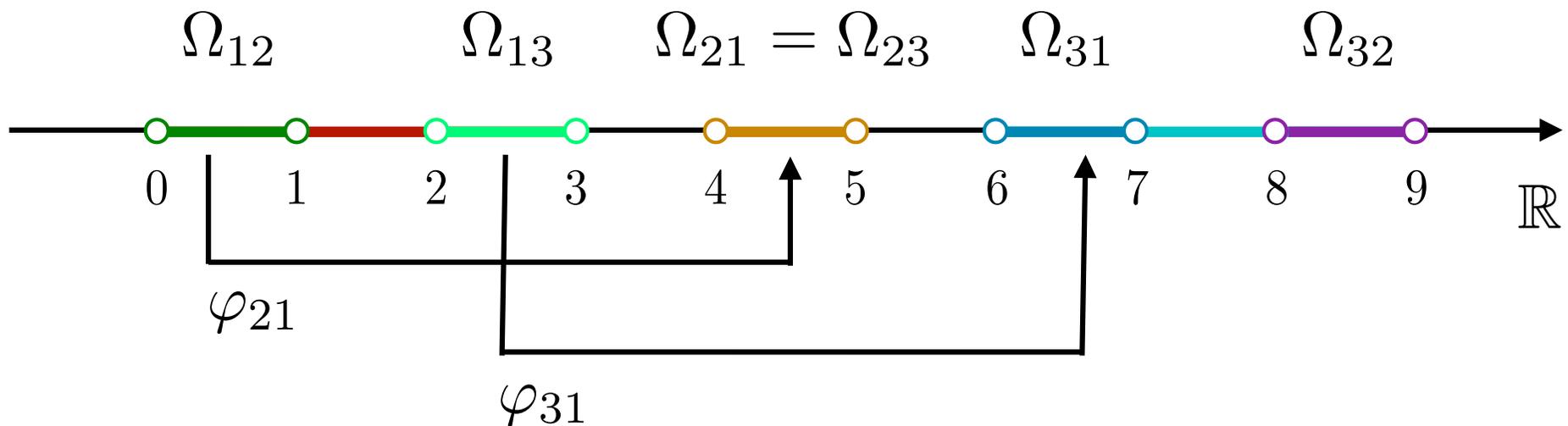
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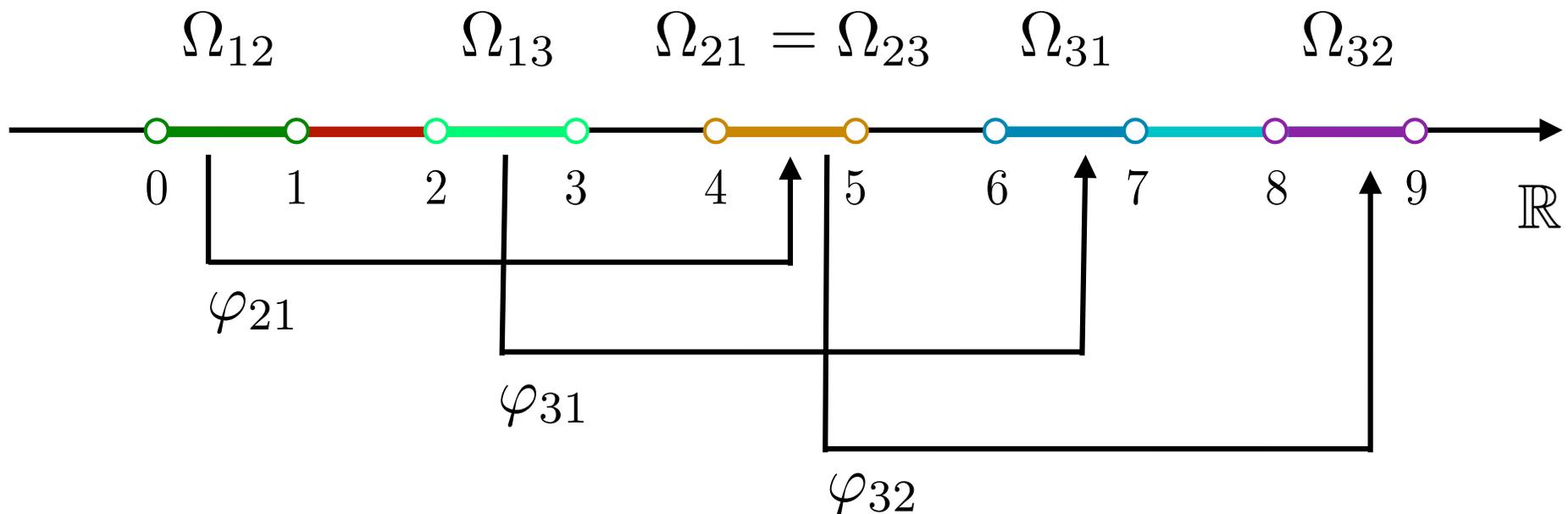
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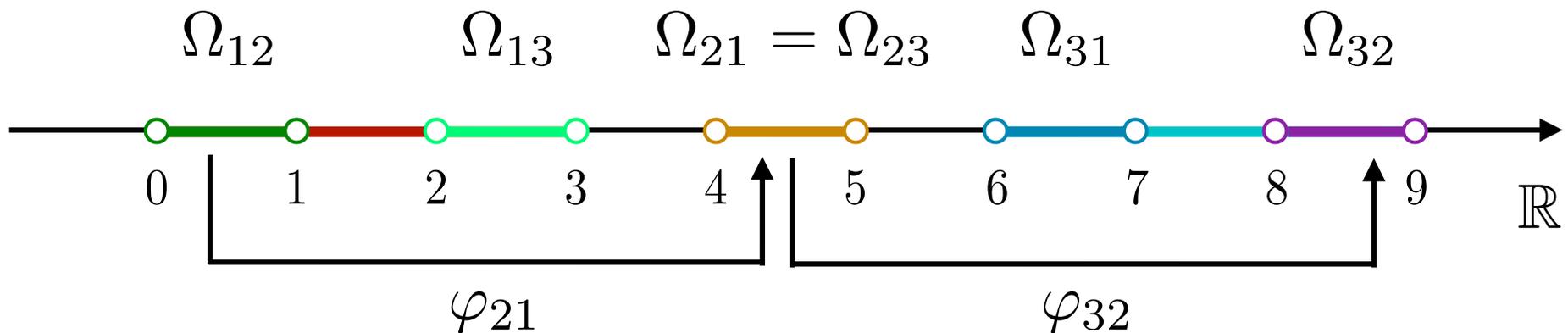
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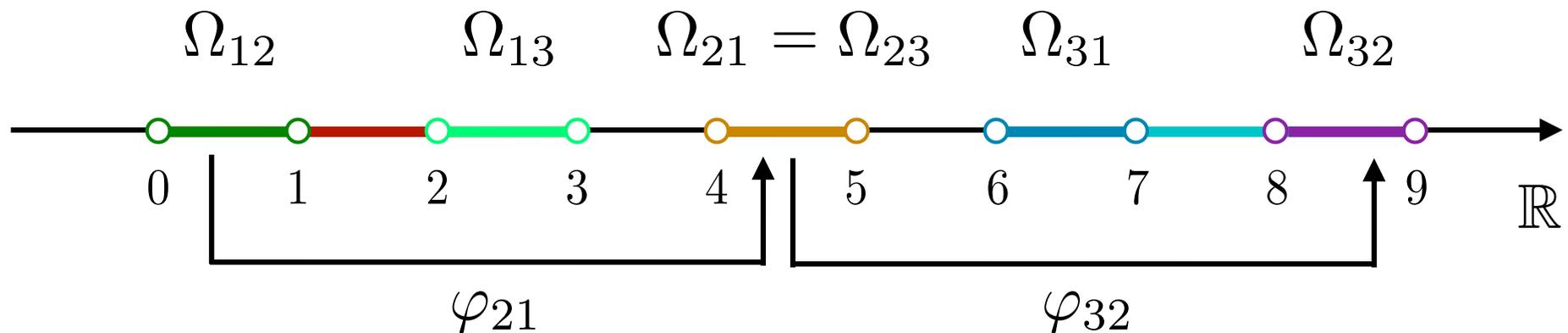
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Note that

$$\Omega_{21} \cap \Omega_{23} = \Omega_2 = (4, 5) \neq \emptyset,$$

but

$$\varphi_{21}^{-1}(\Omega_{21} \cap \Omega_{23}) = (0, 1) \not\subseteq (2, 3) = \Omega_{13}.$$

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So, the statement

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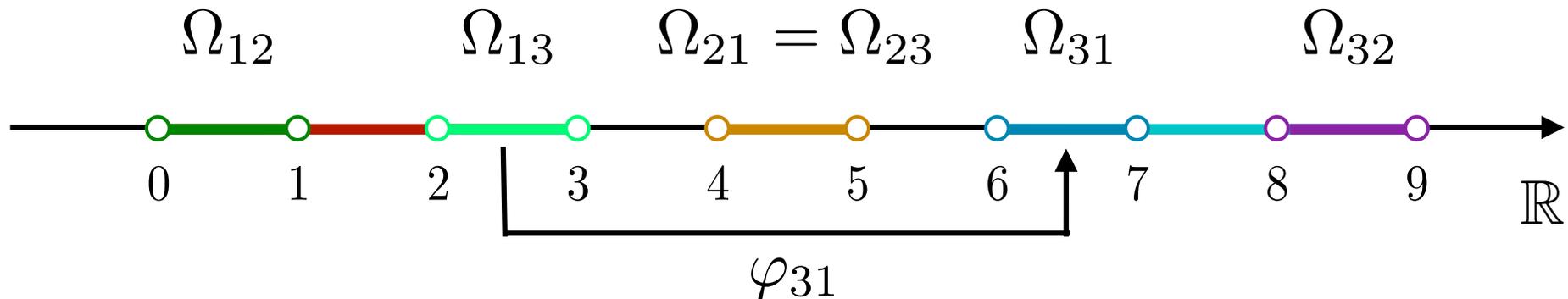
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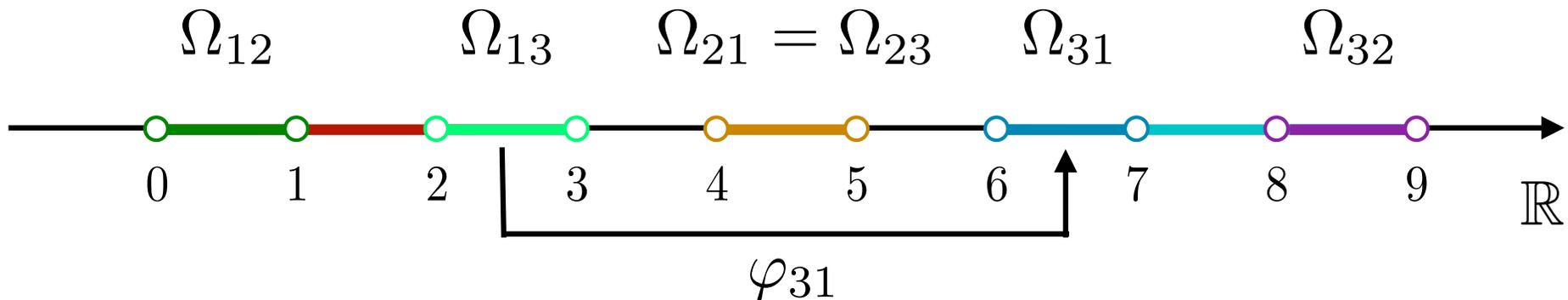


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It turns out that  $\varphi_{31}$  is *undefined* in  $\varphi_{21}^{-1}(\Omega_{21} \cap \Omega_{23})$ .

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*Given a set of gluing data,  $\mathcal{G}$ , can we build a manifold from it?*

- Indeed, such a manifold is built by a **quotient construction**.
- We form the disjoint union of the  $\Omega_i$  and we identify  $\Omega_{ij}$  with  $\Omega_{ji}$  using  $\varphi_{ji}$ , an equivalence relation,  $\sim$ . We form the quotient

$$M_{\mathcal{G}} = \left( \coprod_i \Omega_i \right) / \sim, .$$

# Parametric Pseudo-Manifolds

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**Theorem 1** [Gallier, Siqueira, and Xu, 2008]

For every set of gluing data,

$$\mathcal{G} = \left( (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K} \right),$$

there is a  $n$ -dimensional  $C^k$  manifold,  $M_{\mathcal{G}}$ , whose transition functions are the  $\varphi_{ji}$ 's.

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A condition on the gluing data is needed to make sure that  $M_{\mathcal{G}}$  is Hausdorff:

- (4) For every pair  $(i, j) \in K$ , with  $i \neq j$ , for every  $x \in \partial(\Omega_{ij}) \cap \Omega_i$  and every  $y \in \partial(\Omega_{ji}) \cap \Omega_j$ , there are open balls,  $V_x$  and  $V_y$  centered at  $x$  and  $y$ , so that no point of  $V_y \cap \Omega_{ji}$  is the image of any point of  $V_x \cap \Omega_{ij}$  by  $\varphi_{ji}$ .

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Theorem 1 is very nice, but ...

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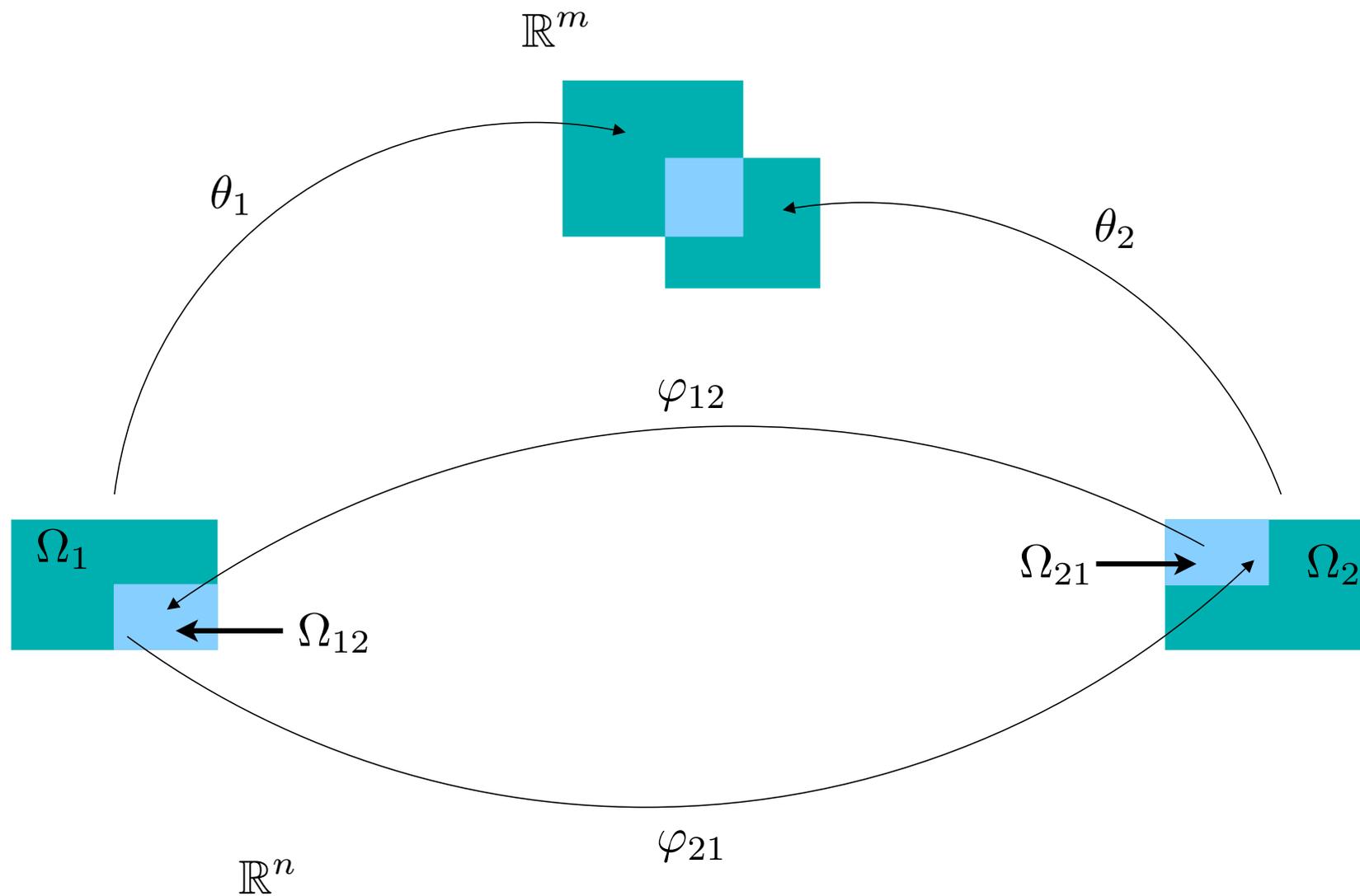
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Let us first formalize our notion of “concreteness”.

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## Big Picture



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A **parametric  $C^k$  pseudo-manifold of dimension  $n$  in  $\mathbb{R}^m$**  is a pair

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}),$$

such that  $\mathcal{G} = ((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K})$  is a set of gluing data, for some *finite*  $I$ , and each  $\theta_i$  is a  $C^k$  function,  $\theta_i : \Omega_i \rightarrow \mathbb{R}^m$ , called a **parametrization**, such that the following holds:

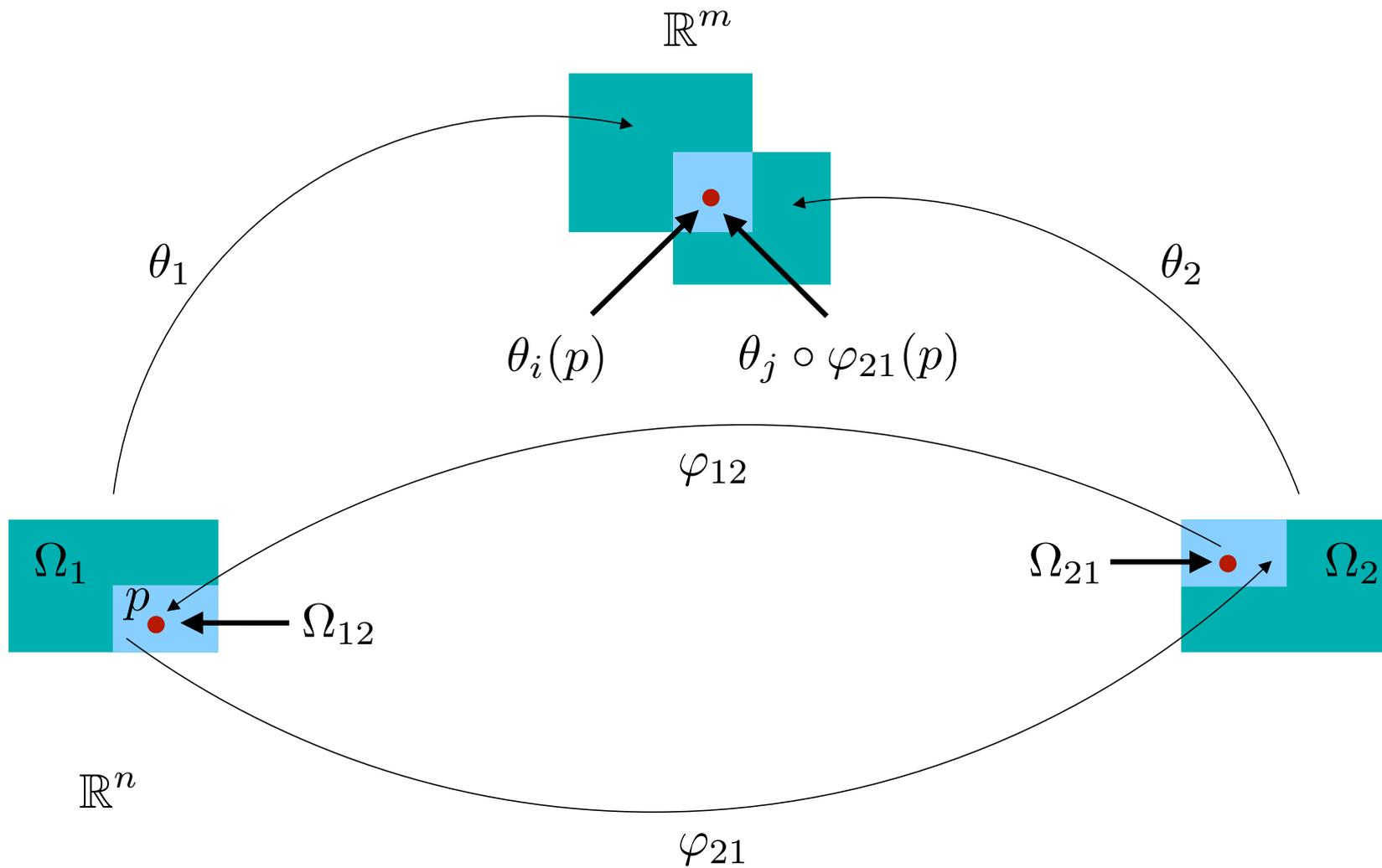
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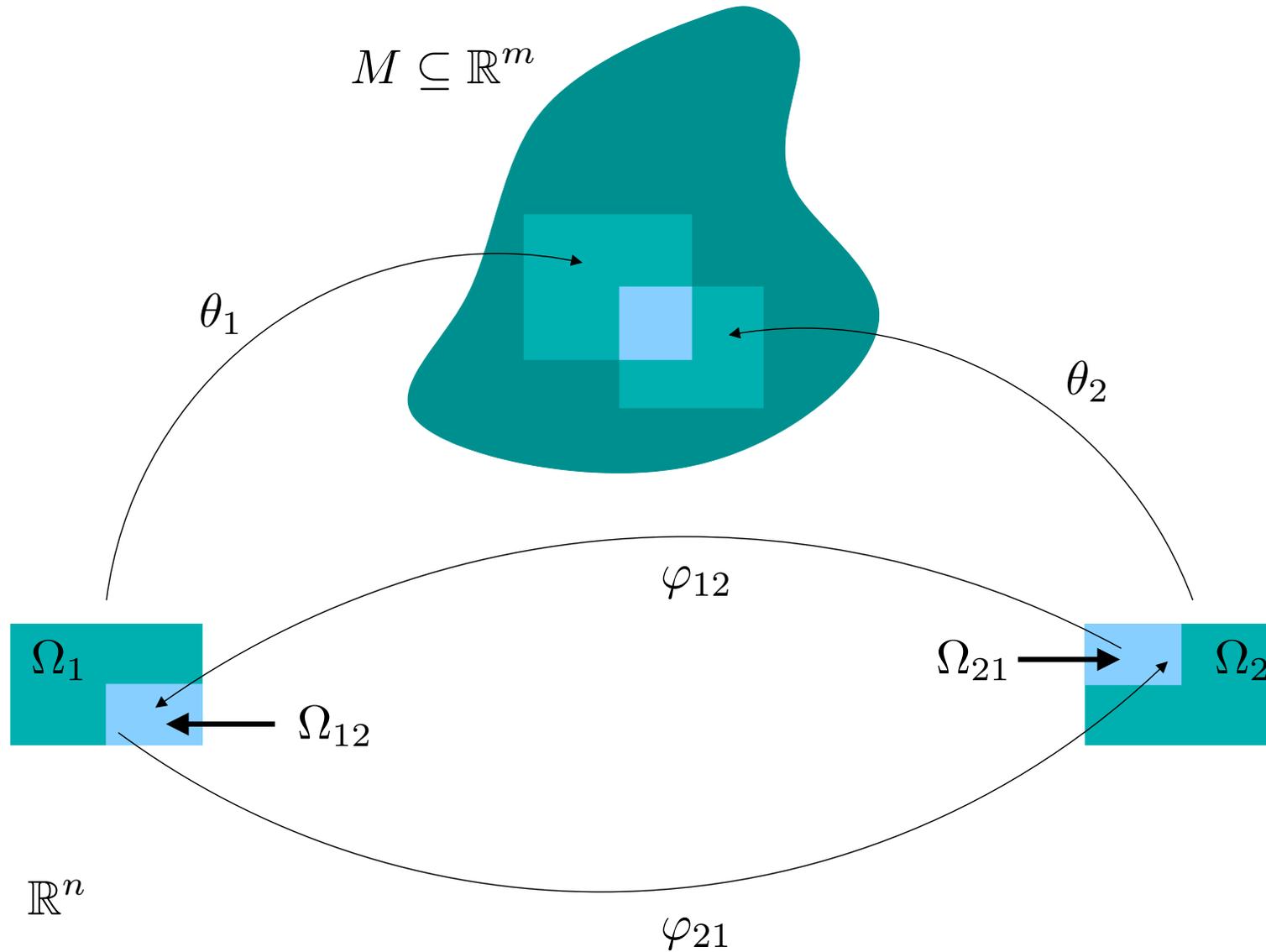
- The subset

$$M = \bigcup_{i \in I} \theta_i(\Omega_i)$$

of  $\mathbb{R}^m$  is called the **image** of the parametric pseudo-manifold.

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- When  $m = 3$  and  $n = 2$ , we say that  $\mathcal{M}$  is a **parametric pseudo-surface**.
- *Under certain conditions* (which we shall see in the next slide), the image of a parametric pseudo-surface is a surface in  $\mathbb{R}^3$ .

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(C') For all  $(i, j) \in K$ ,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}).$$

(C'') For all  $(i, j) \notin K$ ,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset.$$

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# Conclusions

- We can *build* a parametric pseudo-manifold (PPM) from a set of gluing data and, *under certain conditions*, the image of a PPM can be given the structure of a manifold.
- In the last lecture, we will describe a new constructive approach to define a set of gluing data from a triangle mesh.
- We also describe how to build a parametric  $C^\infty$  pseudo-surface from the set of gluing data. The image of this parametric pseudo-surface approximates the vertices of the mesh.

# Suggested Reading

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- Gallier, J.; *Chapter 3 - Construction of Manifolds from Gluing Data*, Notes on Differential Geometry and Lie Groups.

Download a PDF from the course web page:

<http://w3.impa.br/~lvelho/ppm09>