Constructing Manifolds

Lecture 3 - February 3, 2009 - 1-2 PM

Outline

- Sets of gluing data
- The cocycle condition
- Parametric pseudo-manifolds (PPM's)
- Conclusions

Sets of Gluing Data

Let n and k be integers such that $n \geq 1$ and $k \geq 1$ (or $k = \infty$).

A set of gluing data is a triple

$$\mathcal{G} = \left((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K}\right) ,$$

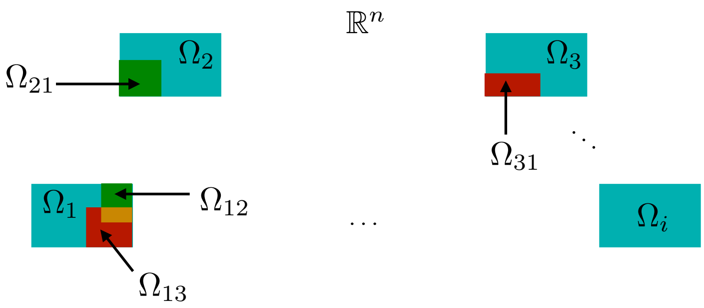
where I and K are countable sets and I is non-empty, satisfying the following three properties:

Sets of Gluing Data

(1) For every $i \in I$, the set Ω_i is a non-empty open subset of $\mathbb{R}^n$ called **parametrization domain**, for short, p-**domain**, and the Ω_i are pairwise disjoint (i.e., $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).

Sets of Gluing Data

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$, and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each non-empty Ω_{ij} (with $i \neq j$) is called **gluing domain**.



Sets of Gluing Data

(3) If we let

$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

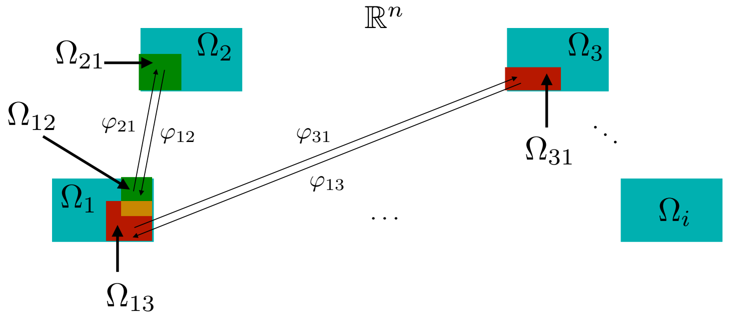
then

$$\varphi_{ji} : \Omega_{ij} \longrightarrow \Omega_{ji}$$

is a C^k bijection for every $(i, j) \in K$, called a **transition function** or **gluing function**.

Sets of Gluing Data

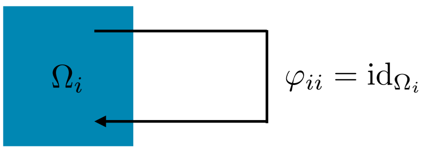
- The transition functions tell us how to glue the p-domains.



Sets of Gluing Data

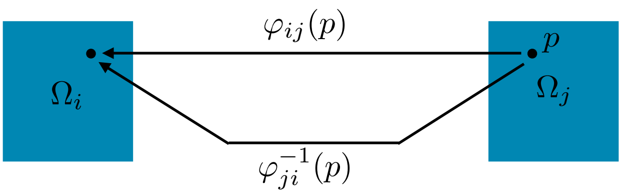
The transition functions must satisfy the following conditions:

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,



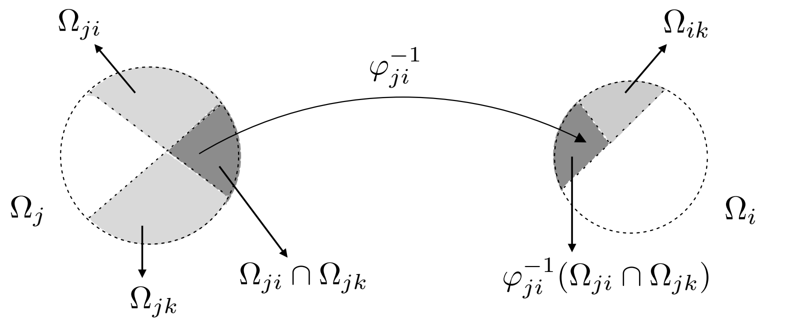
Sets of Gluing Data

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and



Sets of Gluing Data

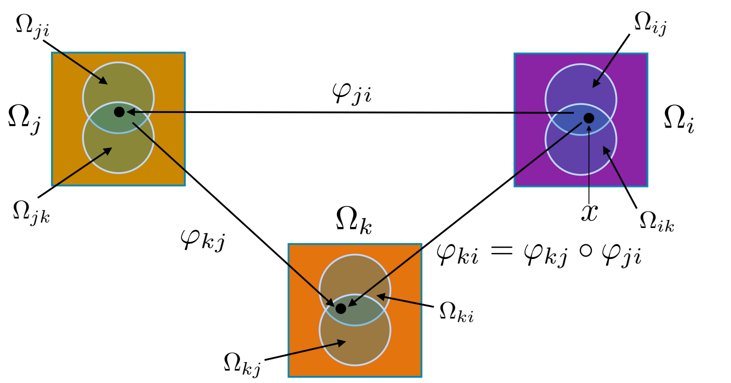
(c) for all i, j, and k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.



The Cocycle Condition

The "evil" cocycle condition

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x), \text{ for all } x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}).$$



The Cocycle Condition

- The cocycle condition implies conditions (a) and (b):

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$, and

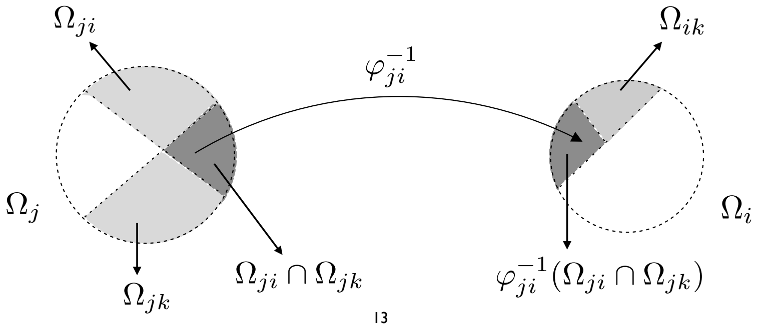
(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$.

The Cocycle Condition

- The statement

$$\text{if } \Omega_{ji} \cap \Omega_{jk} \neq \emptyset \text{ then } \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$$

is **necessary**!

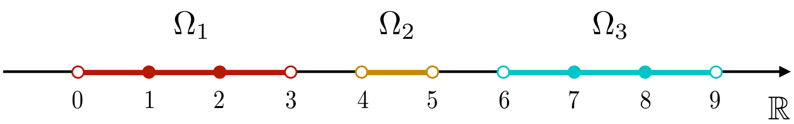


The Cocycle Condition

- Things can go wrong if the condition is false...

Consider the p-domains (i.e., open line intervals)

$$\Omega_1 = (0,3), \quad \Omega_2 = (4,5), \quad \text{and} \quad \Omega_3 = (6,9).$$



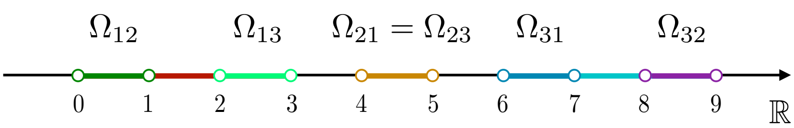
The Cocycle Condition

Consider the gluing domains

$$\Omega_{12} = (0,1) \quad \text{and} \quad \Omega_{13} = (2,3),$$

$$\Omega_{21} = \Omega_{23} = (4,5), \text{ and}$$

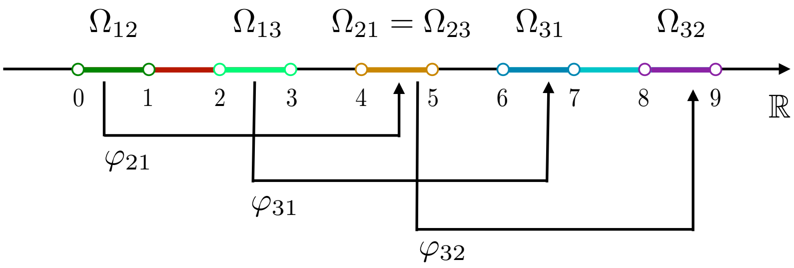
$$\Omega_{32} = (8,9) \quad \text{and} \quad \Omega_{31} = (6,7)\,.$$



The Cocycle Condition

Consider the transition functions:

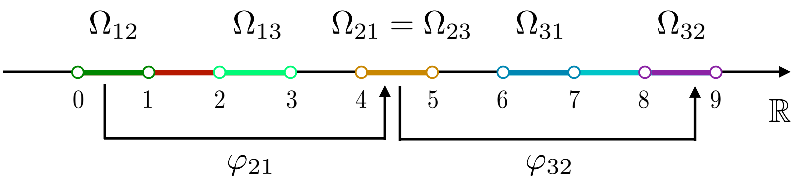
$$\varphi_{21}(x) = x + 4, \quad \varphi_{32}(x) = x + 4, \quad \text{and} \quad \varphi_{31}(x) = x + 4\,.$$



The Cocycle Condition

Obviously,

$$\varphi_{32} \circ \varphi_{21}(x) = x + 8, \ \text{for all } x \in \Omega_{12}\,.$$



Note that

$$\Omega_{21} \cap \Omega_{23} = \Omega_2 = (4, 5) \neq \emptyset\,,$$

but

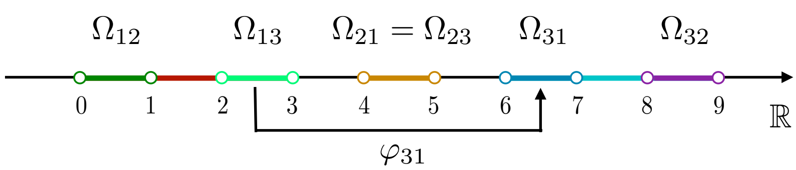
$$\varphi_{21}^{-1}(\Omega_{21} \cap \Omega_{23}) = (0, 1) \not\subseteq (2, 3) = \Omega_{13}\,.$$

The Cocycle Condition

So, the statement

$$\text{if } \Omega_{21} \cap \Omega_{23} \neq \emptyset \text{ then } \varphi_{21}^{-1}(\Omega_{21} \cap \Omega_{23}) \subseteq \Omega_{13}$$

is **false**!



It turns out that φ_{31} is *undefined* in $\varphi_{21}^{-1}(\Omega_{21} \cap \Omega_{23})$.

Parametric Pseudo-Manifolds

- The question now becomes:

Given a set of gluing data, $\mathcal{G}$, can we build a manifold from it?

- Indeed, such a manifold is built by a **quotient construction**.
- We form the disjoint union of the Ω_i and we identify Ω_{ij} with Ω_{ji} using φ_{ji}, an equivalence relation, $\sim$. We form the quotient

$$M_{\mathcal{G}} = \left(\coprod_i \Omega_i \right) / \sim, .$$

Parametric Pseudo-Manifolds

Theorem 1 [Gallier, Siqueira, and Xu, 2008]

For every set of gluing data,

$$\mathcal{G} = \left((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K} \right) ,$$

there is a n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the φ_{ji}'s.

Parametric Pseudo-Manifolds

REMARK:

A condition on the gluing data is needed to make sure that $M_{\mathcal{G}}$ is Hausdorff:

(4) For every pair $(i, j) \in K$, with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and every $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji}.

Parametric Pseudo-Manifolds

Theorem 1 is very nice, but ...

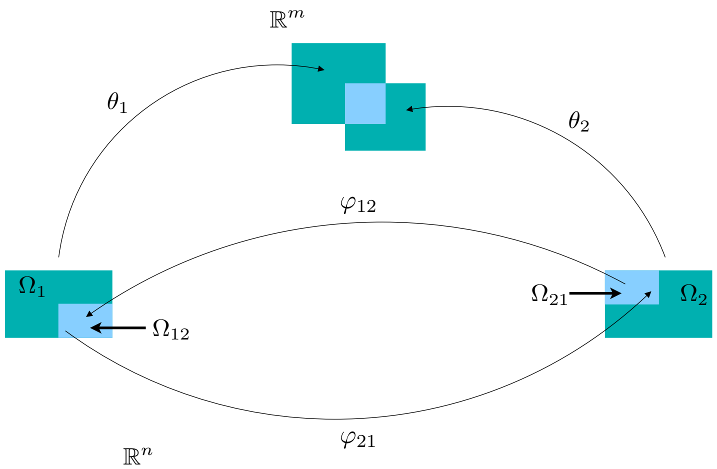
- Our proof is not constructive;
- $M_{\mathcal{G}}$ is an abstract entity, which may not even be compact, orientable, etc.

So, the question that remains is **how** to build a *concrete* manifold.

Let us first formalize our notion of "concreteness".

Parametric Pseudo-Manifolds

Big Picture



Parametric Pseudo-Manifolds

Let n, m, and k be integers, with $m > n \geq 1$ and $k \geq 1$ or $k = \infty$.

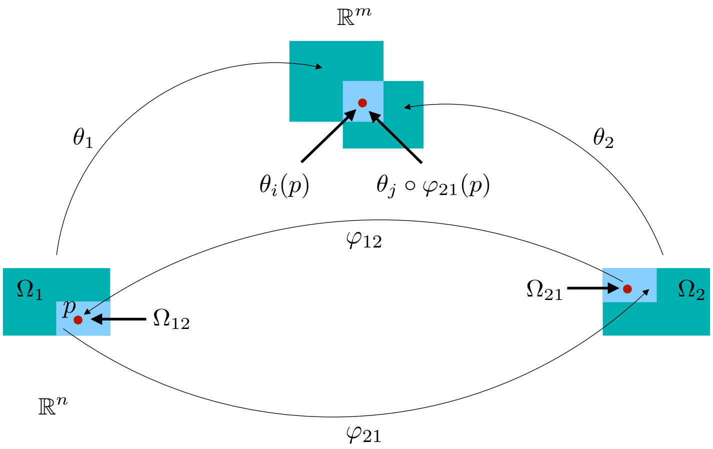
A **parametric C^k pseudo-manifold of dimension n in $\mathbb{R}^m$** is a pair

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}),$$

such that $\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$ is a set of gluing data, for some *finite* I, and each θ_i is a C^k function, $\theta_i : \Omega_i \rightarrow \mathbb{R}^m$, called a **parametrization**, such that the following holds:

Parametric Pseudo-Manifolds

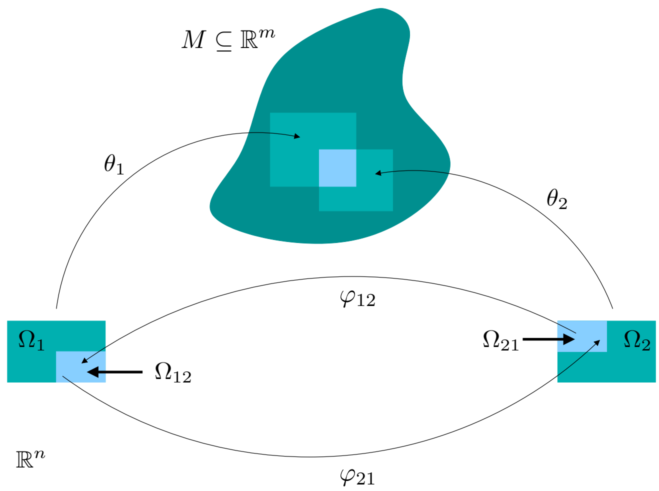
(C) For all $(i,j) \in K$, we have $\theta_i = \theta_j \circ \varphi_{ji}$.



Parametric Pseudo-Manifolds

- The subset
$$M = \bigcup_{i \in I} \theta_i(\Omega_i)$$
of $\mathbb{R}^m$ is called the **image** of the parametric pseudo-manifold.

Parametric Pseudo-Manifolds



Parametric Pseudo-Manifolds

- When $m = 3$ and $n = 2$, we say that $\mathcal{M}$ is a **parametric pseudo-surface**.
- *Under certain conditions* (which we shall see in the next slide), the image of a parametric pseudo-surface is a surface in $\mathbb{R}^3$.

Parametric Pseudo-Manifolds

We proved that M can be given a manifold structure if we require the θ_i's to be bijective and to satisfy the following conditions:

(C') For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}) .$$

(C") For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset .$$

Conclusions

- We can *build* a parametric pseudo-manifold (PPM) from a set of gluing data and, *under certain conditions*, the image of a PPM can be given the structure of a manifold.
- In the last lecture, we will describe a new constructive approach to define a set of gluing data from a triangle mesh.
- We also describe how to build a parametric C^∞ pseudo-surface from the set of gluing data. The image of this parametric pseudo-surface approximates the vertices of the mesh.

Suggested Reading

- Gallier, J.; *Chapter 3 - Construction of Manifolds from Gluing Data*, Notes on Differential Geometry and Lie Groups.

  Download a PDF from the course web page:

  `http://w3.impa.br/~lvelho/ppm09`