Fourier Transform Graphical Analysis: an Approach for Digital Image Processing

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Abstract

Digital image processing area frequently uses Fourier Transform. Even this technique is found in several commercial application softwares dedicated to manipulate images. However, what does this operation really realize in the image? This paper intends to give a graphical interpretation of what is the meaning of applying Fourier Transform to a digital image in order to motivate professionals who work wih this technique so as to have a better understanding, and consequently, extract more possibilities from this powerful tool, when processing a digital image.

1. Introduction

Fourier theories were postulated concerning about thermodynamics laws. However, nowadays, they are not restricted to this area of science. The Fourier Transform became a very important tool in mathematics, physics and engineering, and, also, in a very far related areas as: facial recognition [1] and sexual dimorphism [2], among others.

Similarly, Fourier Transform has been also extensively applied to Digital Image Processing area, Nevertheless, there is a complication factor for using this technique in this area: the great number of publications which deal with this subject were designed for scholars and researchers working with mathematics, physics and engineering, having very little bibliography emphasizing this subject under an interpretative and specific point of view related to Digital Image Processing.

2. Definition

The general definition of Fourier theory claims that every mathematical function can be described as a summation of sines and cosines, where each term of this sum, that is, each sine or cosine, has its proper amplitude and frequency. In mathematical notation we have:

$$f(t) = a_0 + 2\sum_{k=1}^{\infty}(a_k \cos\frac{2\pi kt}{T} + b_k sin\frac{2\pi kt}{T}) = \sum_{k=-\infty}^{\infty} F_k e^{i\left(\frac{2\pi kt}{T}\right)} \quad (1)$$

Where a_0,a_k,b_k,F_k are constants related to the signal amplitudes, t is time, T is period.

A graphical representation of a signal described as a combination of sines having different frequencies (according Eq.1), is exemplified, using some values to illustrate, in Eq.2 and pictured in Fig.1.

$$f(t) = 10\,\mathrm{sen}(2\pi t) + 6\,\mathrm{sen}(10\pi t) + 0{,}8\,\mathrm{sen}(40\pi t) \quad (2)$$

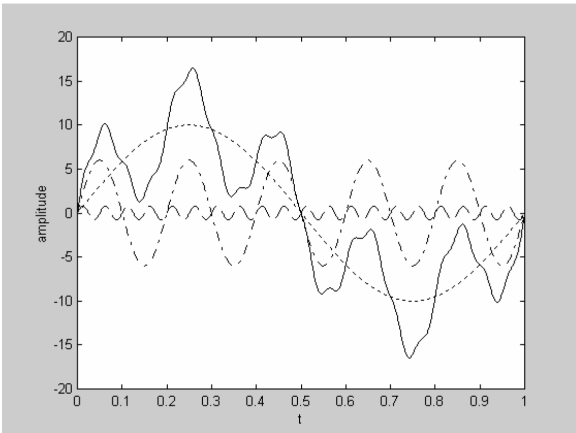

Figure 1. Composed signal f(t) in continuous trace.

Another characteristic of Fourier Transform is that it has an inverse function so it is possible to return back to the original signal as we know the all the representative sines and cosines of the original function.

3. Discrete Fourier Transform applied to digital image processing

The Discrete Fourier Transform (DFT) is a basic operation used to transform an ordered sequence of data samples, usually from the time domain, into the frequency domain, so that spectral information about the sequence can become known explicitly.

In the specific case of digital image processing, the sequence of data samples is ordered according the image space domain, that is, according the pixel position in the image.

Assuming initially an image encompassing only one line containing N pixels (samples), the data sequence with N samples of a signal x_k is given by:

$$x_k = [x_1, x_2, x_3, ..., x_N];\ k = 1,2,...,N \qquad (3)$$

Where k is an index indicating the pixel position in the original line-image.

The DFT of x_k consists of N samples given by:

$$X_m = [X_1, X_2, ...X_N]\ ;\ m = 1,2, ...,N \qquad (4)$$

Where m designates the frequency of each component X_m .

The relation between x_k and X_m can be expressed by:

$$X_m = \sum_{k=1}^{N} x_k e^{-i(2\pi mk/N)}; m = 1,...,N \qquad (5)$$

The function $e^{-i(2\pi mk/N)}$ is a complex and periodical sine/cosine curve. If we consider it as a function of k (pixel position index in the original line-image) for each determined m, then its period is N/m (or its frequency is m/N).

In the DFT, thus, X_m is a summed product (which means correlation) between the data sequence x_k and a sequence of sines/cosines having a period of N/m (or frequency m/N). The more similar is the data sequence with the sine/cosine curve set, the greater the value of $|X_m|$. As a real image is a two-dimensional function, for a complete image the Two-dimensional Fourier Transform must be used. To use it, it is necessary to apply the Fourier Transform to the lines of the images and afterwards to the columns or vice versa.

The mathematical representation of applying the Two-dimensional Fourier Transform to an image x_{jk} (j = line and k = column) results in an image X_{mn} (m = line and n = column) so that:

$$X_{mn} = \sum_{j=1}^{M}\sum_{k=1}^{N} x_{jk} e^{-i(2\pi mk/N + 2\pi mj/N)}; 1 \le m < M; 1 \le n < N \qquad (6)$$

Each X_{mn} generated by DFT is the contribution of a sequence of sines/cosines having frequency equal to m/N horizontally and a sequence of sines/cosines having frequency equal to n/N vertically for all the lines and columns of the image. An interest way of represent the exponential factor in Eq.6 can be a matrix (EXP) where each term in the line represents the exponential relating to the sine/cosine curve angles, and in the column each term represents the angle multiplied by a constant dependent on the line number. A didactic way of representing this matrix is showed in Fig.2. As x_{jk} is a matrix, to multiply x_{jk} by EXP (also a matrix), means realize the first summation in the Eq.6. Afterwards, the result must be transposed and multiplied again by EXP, realizing the second summation in the Eq.6**.** At the end, the result must be transposed again to return to the original position. Each term multiplied tries to find similarity between the function matrix in the image space domain (x_{jk}) and the functions used as "models" ($e^{-i(2\pi mk/N)}$), that is, the sines and cosines represented by the exponential term.

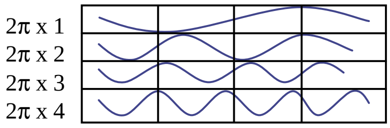

Figure 2. EXP Matriz related to Eq.6

So, it is possible to visualize that what Discrete Fourier Transform is looking for is to find the frequencies which composes the function in the image space domain, sweeping the image with all the candidate functions, analysing the similarity between them. Fig.3 shows the curves of Fig.1, but this time not in the time domain. Fig.3 shows the same curves however in the image space domain, as a sequence of pixels with amplitudes related to gray levels. In this example the image has only one line to facilitate. Line 1 is the original line-image function, composed by the summation of sine curves showed in line 2, 3 and 4 as the first, second and third terms of Eq.2, respectively.

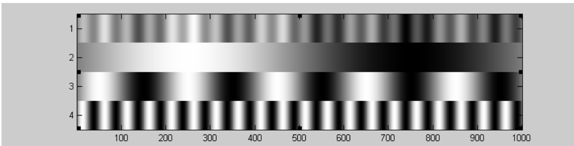

Figure 3. Image representation of Fig.1

4. Conclusion

With the aid of the graphical interpretation and the given example, it is possible understand how the Fourier Transform comes to its results. Future work intends to use the same concept to several two-dimensional images in order to improve FFT undertanding and application when analysing an image. This argumentation allows explore the possibility of alter the functions used as "models" to represent the image in the frequency domain. Actually, another technique explores precisely this possibility – Wavelet Transform.

5. References

[1] Cesar Junior, R. M. ; Campos, T. E., Improved Face X Non-Face Discrimination Using Fourier Descriptors Through Feature Selection. In: 13th SIBGRAPI, Gramado. Proceedings of 13th SIBGRAPI, 2000. p. 28-35, 2000.

[2] Lestrel, P. E. ; Cesar Junior, R. M. ; Takahashi, O. ; Kanazawa, E., Sexual Dimorphism in the Japanese Cranial Base: A Fourier-Wavelet Representation. American Journal of Physical Anthropology, Wiley Interscience, 2005.