# Implicit Models

- Background
- Multiscale Framework
- Piecewise Implicit Representation
- Generating an Implicit Function
- Examples

# Basic Concepts

- Implicit Model $\quad \mathcal{I} = (f, c)$
  - Implicit Function: $f : U \subset \mathbf{R}^n \rightarrow \mathbf{R}$
  - Value: $c \in \mathbf{R}$, constant

$$X = \{(x) \ : \ f(x) \leq c\}$$

- Definition of Geometry
  - Surfaces: $\mathcal{S} = f^{-1}(c)$
  - Solids: $\mathcal{O} = f^{-1}((-\infty, c])$
- Gradient Field

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right).$$

- Characteristic Function
  (point membership)

$$\chi_{\mathcal{O}}(x) = \begin{cases} 1 & \text{if } f(x) \leq c \\ 0 & \text{otherwise} \end{cases}$$

# Multiresolution Implicit Models

- Functions of the Ambient Space
- Wavelet Models
  - Basis: 7 different Wavelet Functions
  - Not Suitable for Synthesis
- Surflet Models
  - Oriented Wavelets (redundant)
  - Not Rigorous
- Multiscale Models
  - Basis: 1 Scaling Function
  - Appropriate for Synthesis

# Multiscale B-spline Representation

- Piecewise Description
  - Simple Elements
  - Adapted to $f$ (variations)
  - Localized (space and frequency)

Representation

Linear Expansion over a Multiscale Dictionary $\mathcal{D}$

$$f(x) = \sum_i a_i \phi_{j_i,k_i}(x)$$

where

$$a_i = < f, \tilde{\phi}_{j_i,k_i} >$$

- "Blobby" Model
  - Hierarchical
  - Structured

# Definitions

- $\mathcal{D}$ generated by a single function $\phi$

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$

  $\phi_j$ basis of the approximation spaces $V_j$

$$V_j = \text{span}\{\phi_{j,k} : k \in \mathbf{Z}\}$$

- Relation with Wavelets

$$W_j = V_{j+1} \ominus V_j$$

$$\bigoplus_j W_j = L^2(\mathbf{R}^n)$$

- Computation

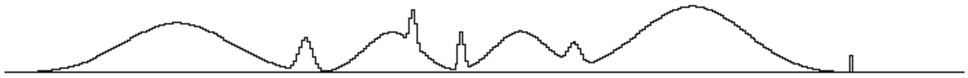
$$A_{j+1} = \mathbf{P}_{V_{j+1}}(\mathbf{P}_{W_j}(f))$$

# Method

1. Start with $\{c\}_0 = < f, \widetilde{\phi}_{0,k} >$
   (i.e. the representation of $f$ relative to $V_0$).
2. Compute the Laplacian decomposition of $f$ generating the pyramid $A_j = \{a_k\}_j$.
3. Construct a compact multiscale representation, keeping only the coefficients $|a_{j,k}| > \epsilon$ and storing them in a spatial hash table.

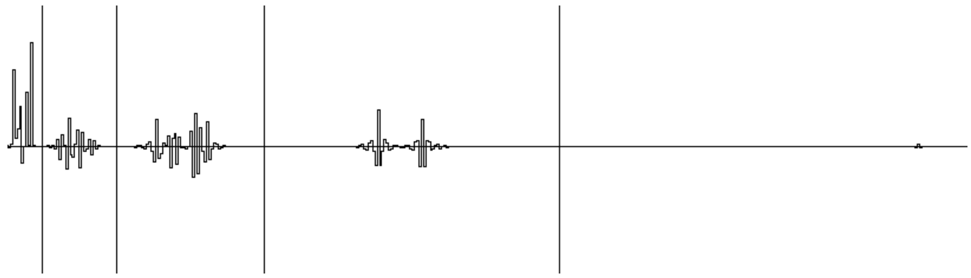
# Example (step 1)

1. Start with $\{c\}_0 = < f, \tilde{\phi}_{0,k} >$
   (i.e. the representation of $f$ relative to $V_0$).
2. 
3. 



# Example (step 2)

1.
2. Compute the Laplacian decomposition of $f$ generating the pyramid $A_j = \{a_k\}_j$.
3.

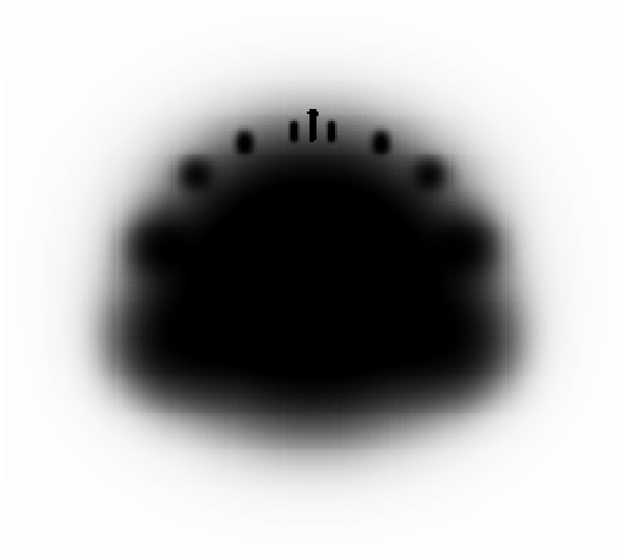


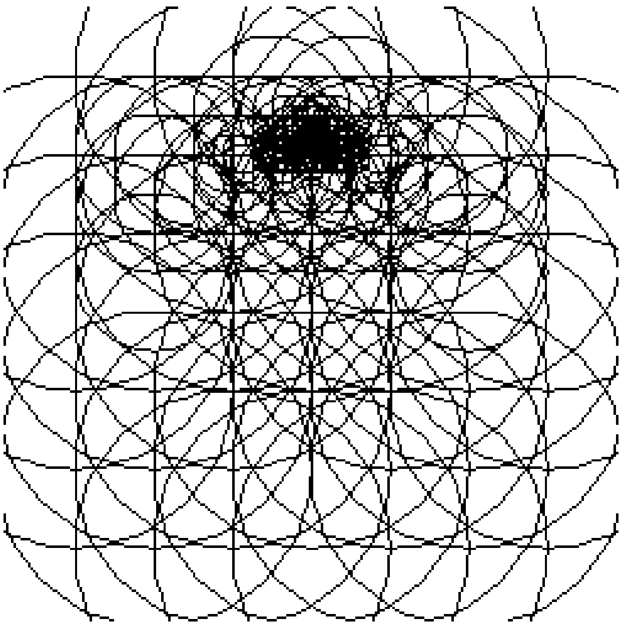
$A_0$ $A_1$ $A_2$ $A_3$ $A_4$

# Example (step 3)

1.
2.
3. Construct a compact multiscale representation, keeping only the coefficients $|a_{j,k}| > \epsilon$ and storing them in a spatial hash table.

# 2D Example



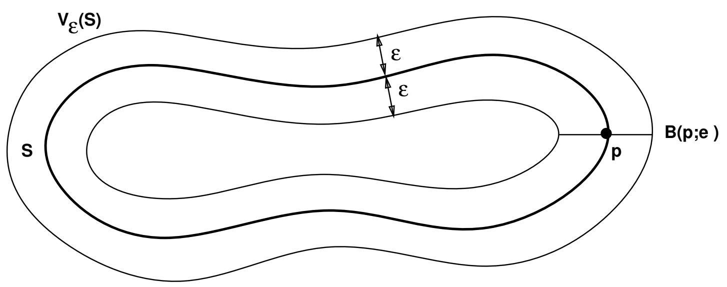


# Smooth Implicit Functions

Shape information near $\mathcal{S}$

- algebraic distance
- "parallel" surfaces

- Tubular Neighborhoods



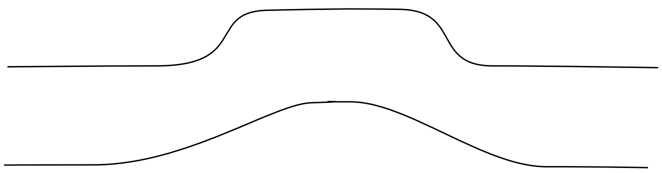

Implications

- Modeling
  - blending
  - offsets
- Computation
  - space subdivision
  - root finding

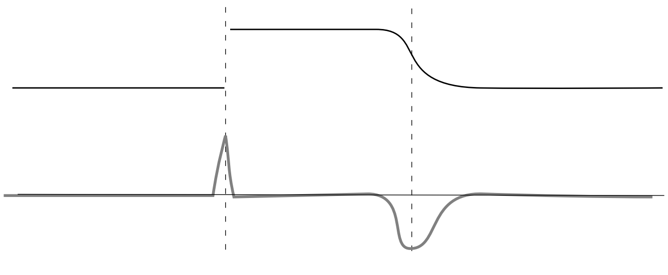
# Multiscale Edge Detection

- Step Edge: (discontinuity)

- Multiscale Edges: $f_s = f * \theta_s$



- Edge Detection with Wavelets: $\psi = \frac{d\theta(x)}{dx}$



$$(T_\psi f)(s, x) = \frac{d}{dx}(f * \theta_s)(x)$$

# Multiscale Edge Synthesis

- Evolution of Wavelet Maxima across Scales
  - Step Edge at $x$

$$|(T_\psi f)(2^j, x)| = C$$

  - Smooth Edge at $x$

$$|(T_\psi f)(2^j, x)| = K 2^j$$

- Reconstruction from Wavelet Maxima
  Alternating Projections Algorithm

$*$ Generate New Maxima Values

Multiscale Edges
$\updownarrow$
Variable Size Neighborhoods

# Method

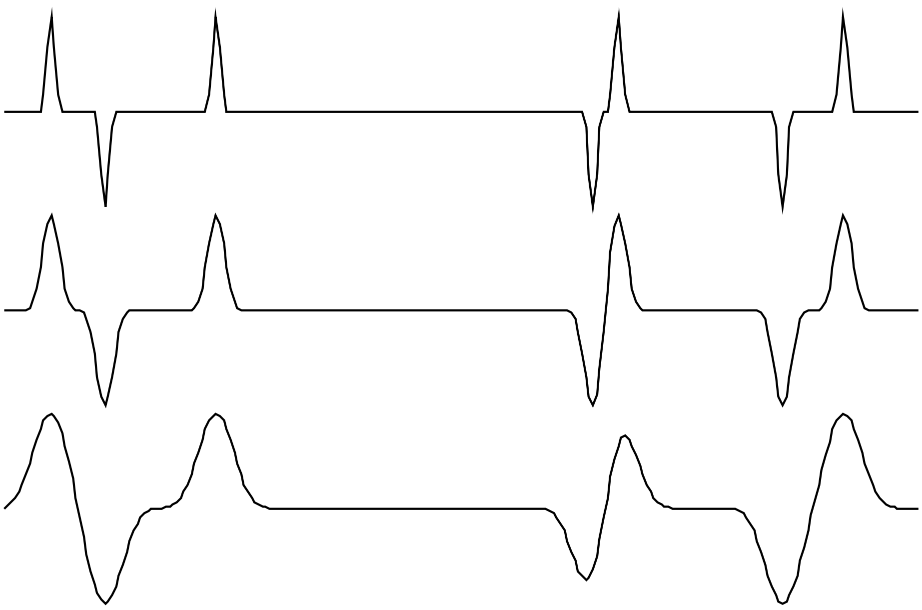
1. Start with $\chi_{\mathcal{O}} : U \to \{0, 1\}$, of a solid $\mathcal{O}$, whose boundary is $\mathcal{S}$.
2. Compute $(T_\psi \chi_{\mathcal{O}})(2^j, x)$ of $\chi_{\mathcal{O}}$ at scales $2^j$, $j = 1, \ldots, J$.
3. Identify the local maxima of $|(T_\psi \chi_{\mathcal{O}})(2^1, x)|$.
4. For each edge point of $\chi_{\mathcal{O}}$ at scale $1$, derive new wavelet maxima values for scales $2^j, j = 1, \ldots, J$.
5. Generate a smooth function $f$ from the new maxima values.

# Example (step 1)

1. Start with $\chi_{\mathcal{O}} : U \to \{0, 1\}$, of a solid $\mathcal{O}$, whose boundary is $\mathcal{S}$.
2.
3.
4.
5.

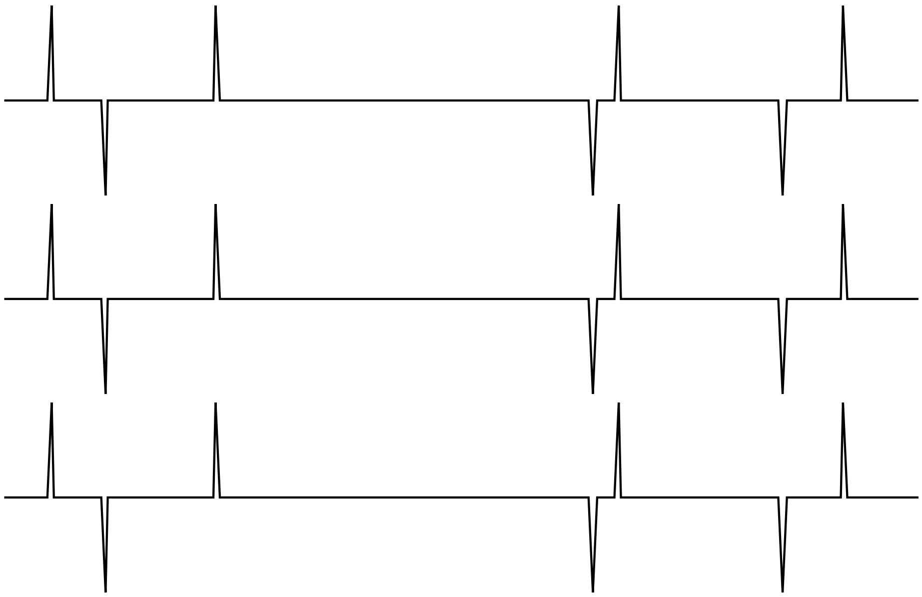
# Example (step 2)

1.
2. Compute $(T_\psi \chi_\mathcal{O})(2^j, x)$ of $\chi_\mathcal{O}$ at scales $2^j$, $j = 1, \ldots, J$.
3.
4.
5.



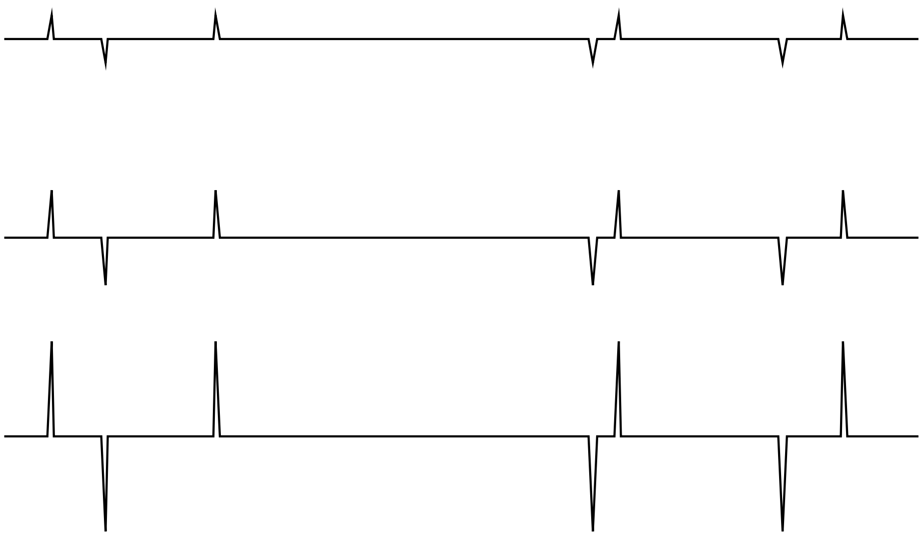
# Example (step 3)

1.
2.
3. Identify the local maxima of $|(T_\psi \chi_{\mathcal{O}})(2^1, x)|$.
4.
5.



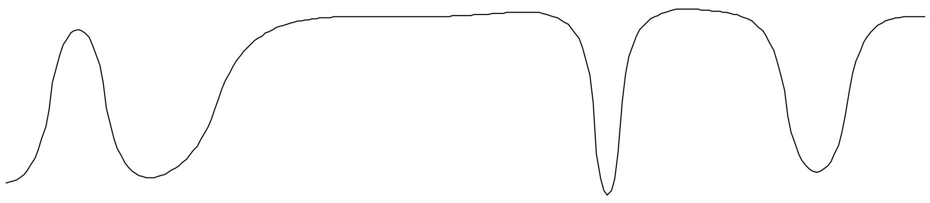
# Example (step 4)

1.
2.
3.
4. For each edge point of $\chi_{\mathcal{O}}$ at scale $1$, derive new wavelet maxima values for scales $2^j, j = 1, \ldots, J$.
5.

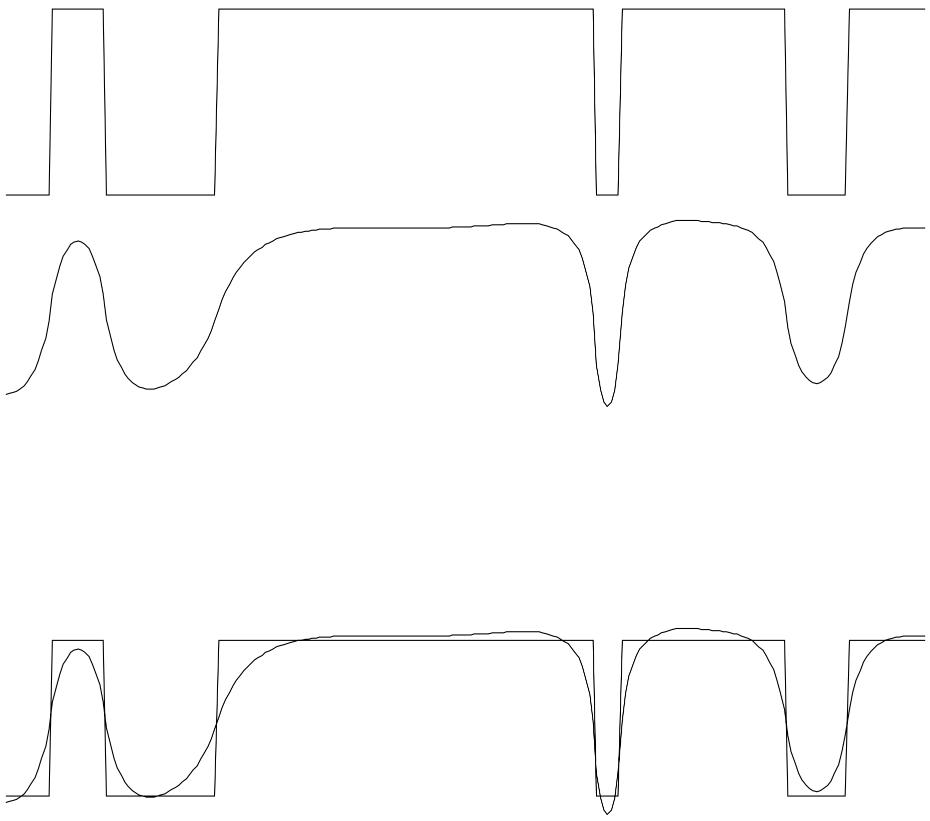


# Example (step 5)

1.
2.
3.
4.
5. Generate a smooth function $f$ from the new maxima values.



# Comparison



# Conversion

Parametric $\leftrightarrow$ Implicit

(complementary descriptions)

- Modeling
- Rendering
- Others

- Exact Conversion

  symbolic methods
  - not possible, or
  - not practical
- Approximate Conversion

  numerical methods
  - often sufficient
  - more flexible

# Multiscale Framework for Conversion

- Implicit Model
  - Multiscale
  - Piecewise
- Goals
  - General
  - Adapted
  - Variable Precision

Process

1. Generation of Implicit Function
   (multiscale edge synthesis)
2. Construction of Implicit Representation
   (b-spline multiscale decomposition)

- Wavelets (two types)
  - dyadic Gaussian approximation
  - biorthogonal B-spline

Characteristic Function

$\downarrow$

$\boxed{\text{Multiscale Edge Synthesis}}$

$\downarrow$

Smooth Implicit Function

$\downarrow$

$\boxed{\text{Multiscale Decomposition}}$

$\downarrow$

Hierarchical B-spline Representation

# Conversion Parametric to Implicit

$$\begin{array}{c}\text{Parametric Representation}\\ \downarrow \\ \boxed{\text{Volumetric Rasterization}} \\ \downarrow\end{array}$$

# Constructing the Characteristic Function

- Method
  1. Rasterize the parametric surface
  2. Fill in the enclosed volume

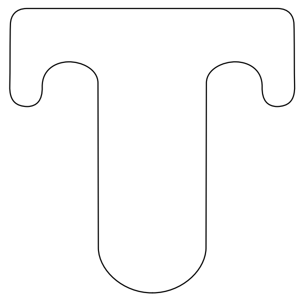
- Issues
  - Resolution

# Results

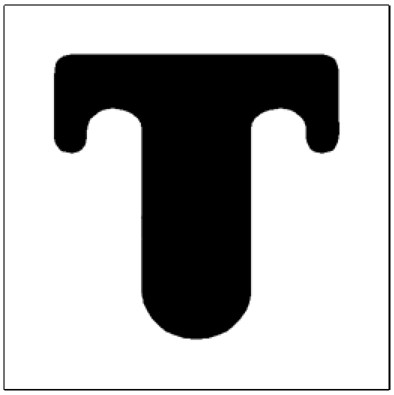
- Examples
  2D (T-Shape)
  3D (Noisy Sphere)

- Pipeline

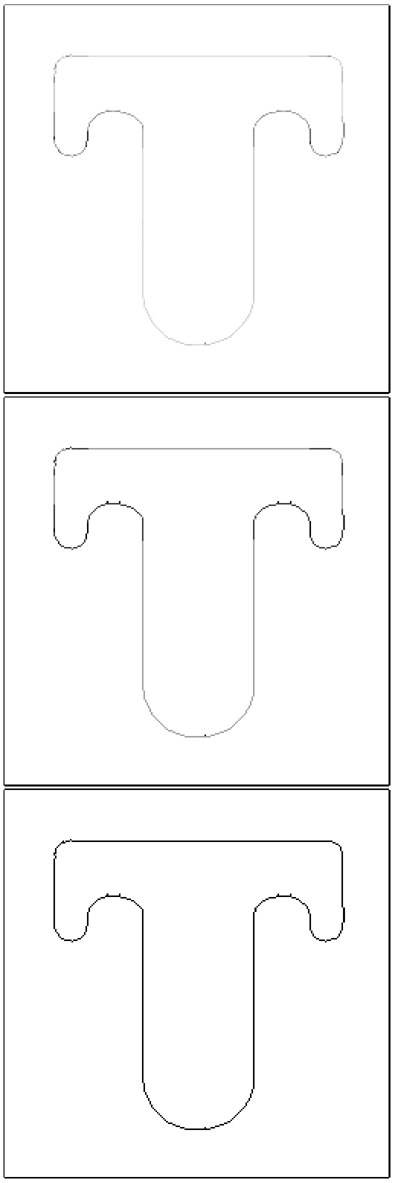
$$\begin{array}{c}\text{Parametric}\\ \downarrow \\ \text{Volumetric}\\ \downarrow \\ \text{Implicit}\end{array}$$



T-Shape in Parametric Form (Bezier curve)

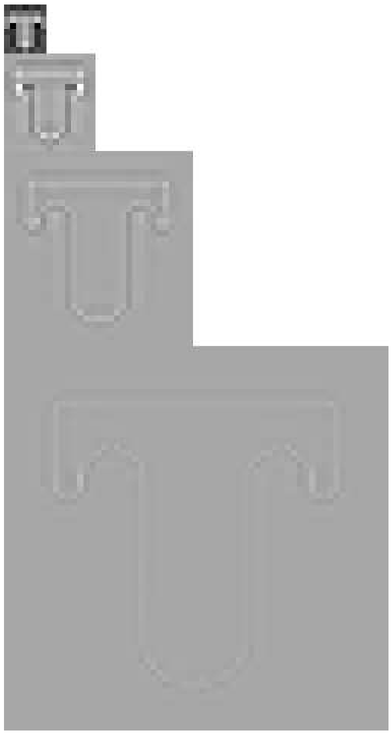


Characteristic Function of the T-Shape

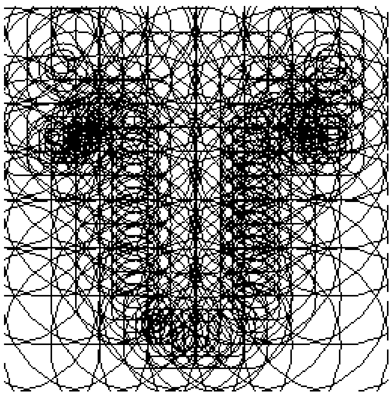


Maxima Values

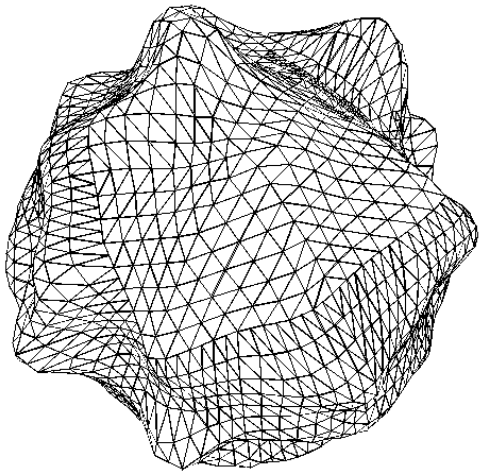
Smooth Implicit Function



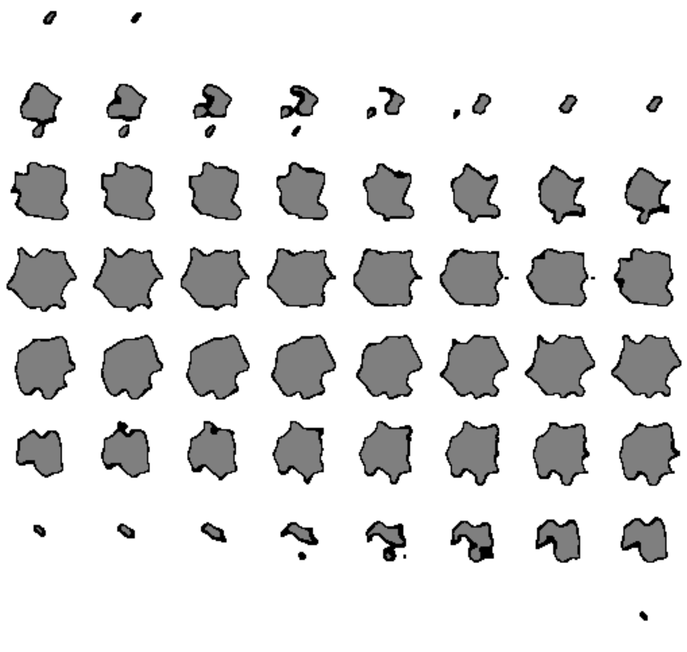
B-spline Pyramid



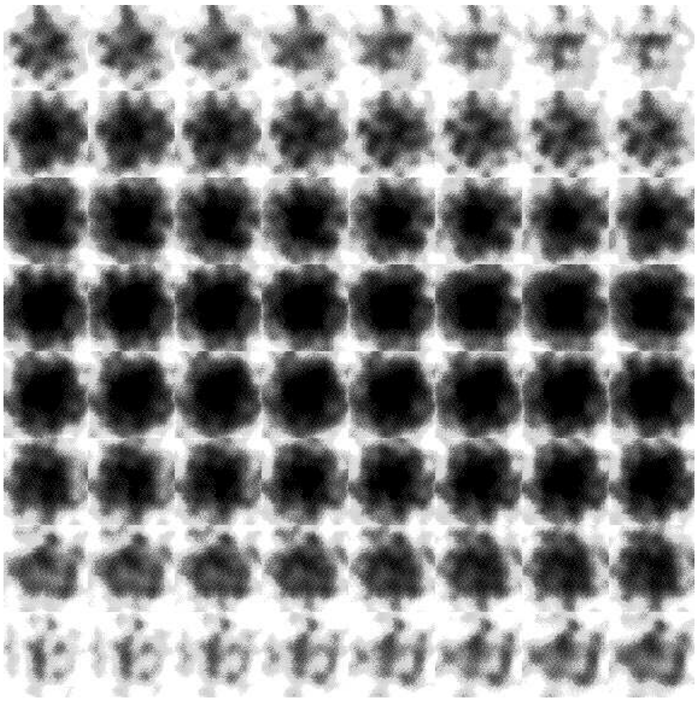
B-spline Functions



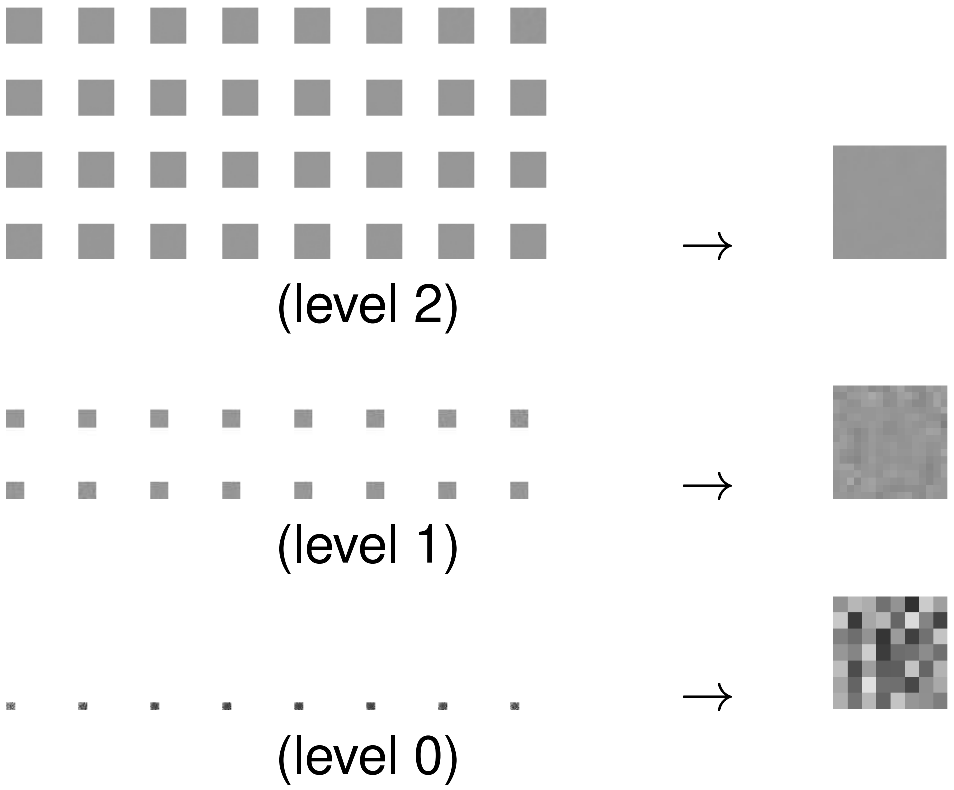
Polygonization of the Noisy Sphere



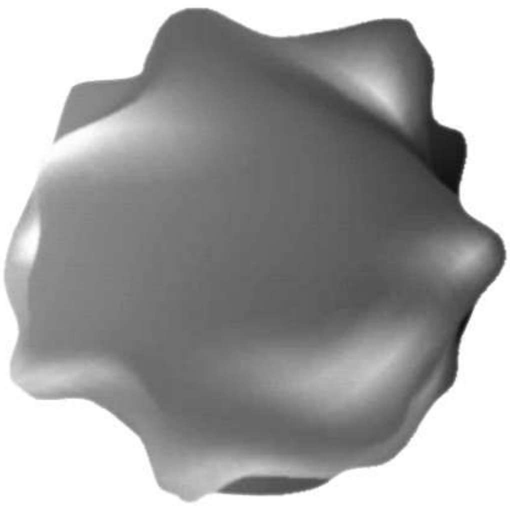
Slices of the Characteristic Function



Slices of the Volume Density Function



B-spline Pyramid and Enlargement of Slices



Noisy Sphere, Raytraced from B-spline Pyramid

# Conclusions

- Multiscale Framework Implicit Models
  - Effective
  - Efficient
- Applications
  - Conversion Explicit $\rightarrow$ Implicit
  - Universal Representation